# UNIT – I- Electrostatics

**Contents**

## INTRODUCTION

### VECTOR ALGEBRA

Vector Algebra is a part of algebra that deals with the theory of vectors and vector spaces.

Most of the physical quantities are either scalar or vector quantities.

### SCALAR QUANTITY:

Scalar is a number that defines magnitude. Hence a scalar quantity is defined as a quantity that has magnitude only. A scalar quantity does not point to any direction i.e. a scalar quantity has no directional component.

For example when we say, the temperature of the room is 30o C, we don‘t specify the direction.

Hence examples of scalar quantities are mass, temperature, volume, speed etc.

A scalar quantity is represented simply by a letter – A, B, T, V, S.

### VECTOR QUANTITY:

A Vector has both a magnitude and a direction. Hence a vector quantity is a quantity that has both magnitude and direction.

Examples of vector quantities are force, displacement, velocity, etc.

$\vec{\mathbf{A}}, \vec{\mathbf{V}}, \vec{\mathbf{B}}, \vec{\mathbf{F}}$

A vector quantity is represented by a letter with an arrow over it or a bold letter.

### UNIT VECTORS:

When a simple vector is divided by its own magnitude, a new vector is created known as the unit vector. A unit vector has a magnitude of one. Hence the name - unit vector.

A unit vector is always used to describe the direction of respective vector.

$$\mathbf{a}_A = \frac{\vec{\mathbf{A}}}{|\mathbf{A}|} \Rightarrow \vec{\mathbf{A}} = |\mathbf{A}|\,\mathbf{a}_A$$

Hence any vector can be written as the product of its magnitude and its unit vector. Unit Vectors along the co-ordinate directions are referred to as the base vectors. For example unit vectors along X, Y and Z directions are ax, ay and az respectively.

### Position Vector / Radius Vector ($\overline{OP}$ ):

A Position Vector / Radius vector define the position of a point(P) in space relative to the origin(O).Hence Position vector is another way to denote a point in space.



$$\overline{OP} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$$

**Displacement Vector**

Displacement Vector is the displacement or the shortest distance from one point to another.

**Vector Multiplication**

When two vectors are multiplied the result is either a scalar or a vector depending on how they are multiplied. The two important types of vector multiplication are:

- Dot Product/Scalar Product (A.B)
- Cross product (A x B)

**1. DOT PRODUCT (A. B):**

Dot product of two vectors A and B is defined as:

$$\bar{A}.\bar{B} = \left|\bar{A}\right| \left|\bar{B}\right| \cos\theta_{AB}$$

Where $\theta_{AB}$ is the angle formed between A and B.
Also $\theta_{AB}$ ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$
The result of A.B is a scalar, hence dot product is also known as Scalar Product.

**Properties of Dot Product:**

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then

   $\bar{A}.\bar{B}$= AxBx + AyBy + AzBz

2. $\bar{A}.\bar{B}$= |A| |B|, if $\cos\theta_{AB}$=1 which means $\theta_{AB} = 0^0$

This shows that A and B are in the same direction or we can also say that A and B are parallel to each other.

3. $\bar{A}.\bar{B}$ = - |A| |B|, if $\cos\theta_{AB}$=-1 which means $\theta_{AB}$ = $180^0$.

This shows that A and B are in the opposite direction or we can also say that A and B are antiparallel to each other.

4. $\bar{A}.\bar{B}$ = 0, if $\cos\theta_{AB}$=0 which means $\theta_{AB}$ = $90^0$.

This shows that A and B are orthogonal or perpendicular to each other.

5. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x.\bar{a}_x = \bar{a}_y.\bar{a}_y = \bar{a}_z.\bar{a}_z = 1$$

$$\bar{a}_x.\bar{a}_y = \bar{a}_y.\bar{a}_z = \bar{a}_z.\bar{a}_x = 0$$

**2. Cross Product (A X B):**

Cross Product of two vectors A and B is given as:

$$\bar{A}X\bar{B} = |\bar{A}|\,|\bar{B}|\,\sin\theta_{AB}\,\bar{a}_N$$

Where $\theta_{AB}$ is the angle formed between A and B and $\bar{a}_N$ is a unit vector normal to both A and B. Also θ ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$

The cross product is an operation between two vectors and the output is also a vector.

**Properties of Cross Product:**

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then,

$$\mathbf{A} * \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix}$$

The resultant vector is always normal to both the vector A and B.

2. $\bar{A}X\bar{B}$ = 0, if $\sin\theta_{AB}$ = 0 which means $\theta_{AB}$ = $0^0$ or $180^0$;
This shows that A and B are either parallel or antiparallel to each other.

63. $\bar{A}X\bar{B} = |\bar{A}|\,|\bar{B}|\,\bar{a}_N$, if $\sin\theta_{AB}$ = 0 which means $\theta_{AB}$ = $90^0$.
This shows that A and B are orthogonal or perpendicular to each other.

4. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x X\,\bar{a}_x = \bar{a}_y\,X\,\bar{a}_y = \bar{a}_z X \bar{a}_z = 0$$

$$\bar{a}_x X\,\bar{a}_y = \bar{a}_z\ ,\bar{a}_y\,X\,\bar{a}_z = \bar{a}_x\ ,\ \bar{a}_z X\,\bar{a}_x = \bar{a}_y$$

## CO-ORDINATE SYSTEMS

Co-Ordinate system is a system of representing points in a space of given dimensions by coordinates, such as the Cartesian coordinate system or the system of celestial longitude and latitude.

In order to describe the spatial variations of the quantities, appropriate coordinate system is required. A point or vector can be represented in a curvilinear coordinate system that may be orthogonal or non-orthogonal. An orthogonal system is one in which the coordinates are mutually perpendicular to each other.

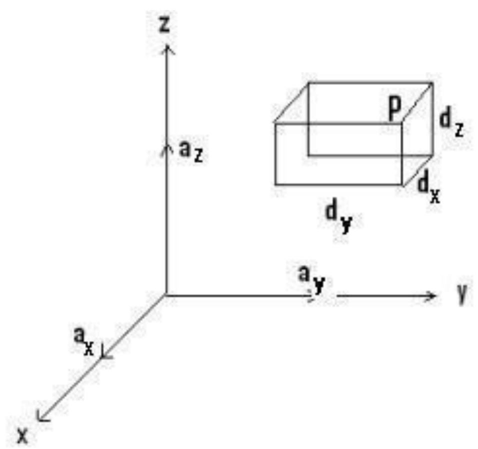
The different co-ordinate system available are:

- Cartesian or Rectangular co-ordinate system.(Example: Cube, Cuboid)
- Circular Cylindrical co-ordinate system.(Example : Cylinder)
- Spherical co-ordinate system. (Example: Sphere)

The choice depends on the geometry of the application.

A set of 3 scalar values that define position and a set of unit vectors that define direction form a co-ordinate system. The 3 scalar values used to define position are called co-ordinates. All coordinates are defined with respect to an arbitrary point called the origin.

### 1. Cartesian Co-ordinate System / Rectangular Co-ordinate System (x,y,z)




A Vector in Cartesian system is represented as (Ax, Ay, Az) Or

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

Where $\bar{a}_x$,$\bar{a}_y$ and $\bar{a}_z$ are the unit vectors in x, y, z direction respectively.

Range of the variables:

It defines the minimum and the maximum value that x, y and z can have in Cartesian system.
**$-\infty \leq x,y,z \leq \infty$**

**Differential Displacement / Differential Length (dl):**

It is given as

$$\overline{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$$

Differential length for a line parallel to x, y and z axis are respectively given as:

dl = $dx\bar{a}_x$---( For a line parallel to x-axis).
dl = $dy\bar{a}_y$ ---( For a line Parallel to y-axis).
dl = $dz\bar{a}_z$ ---( For a line parallel to z-axis).

If there is a wire of length L in z-axis, then the differential length is given as dl = dz az. Similarly if the wire is in y-axis then the differential length is given as dl = dy ay.

**Differential Normal Surface (ds):**

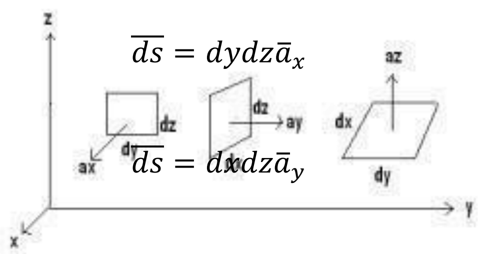
Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.

For the 1st figure,

$$\overline{ds} = dydz\bar{a}_x$$




2nd figure,

$$\overline{ds} = dxdz\bar{a}_y$$

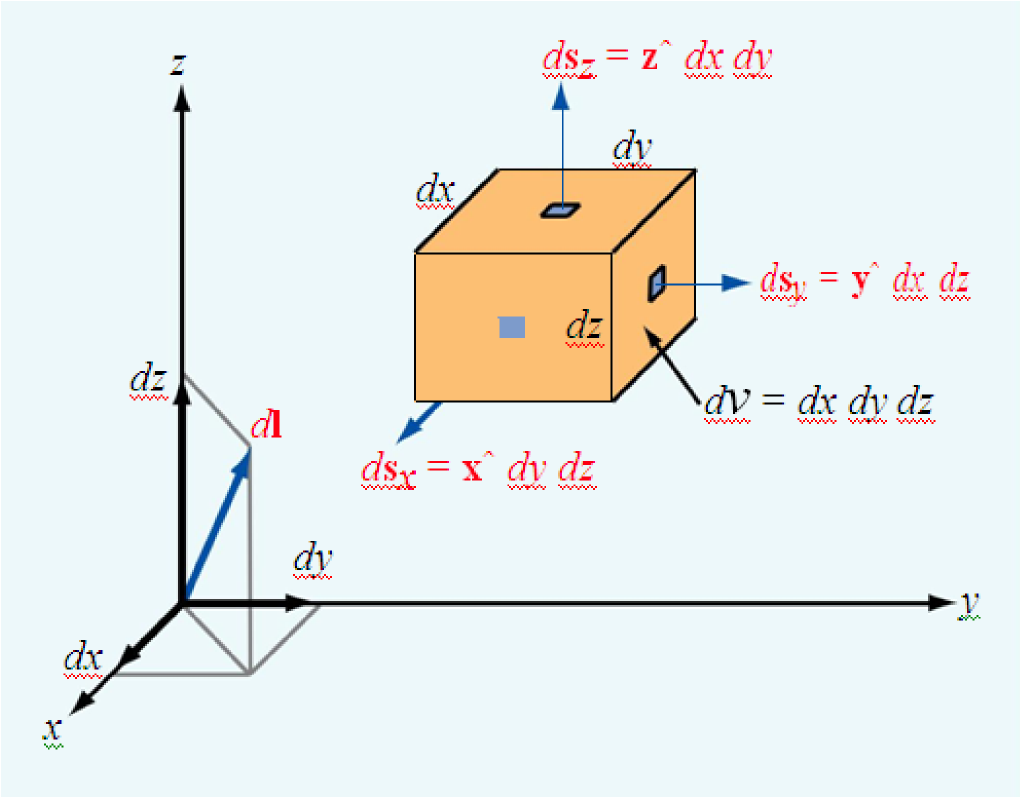
3rd figure,

$$\overline{ds} = dxdy\bar{a}_z$$

**Differential Volume:**

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = dxdydz$$




Differential length, area, and volume in Cartesian coordinates.

## 2. Circular Cylindrical Co-ordinate System

A Vector in Cylindrical system is represented as ($A_r$, $A_\phi$, $A_z$) or

$$\bar{A} = A_r\bar{a}_r + A_\emptyset\bar{a}_\emptyset + A_z\bar{a}_z$$

Where $\bar{a}_r$, $\bar{a}_\emptyset$ and $\bar{a}_z$ are the unit vectors in r, Φ and z directions respectively.

The physical significance of each parameter of cylindrical coordinates:

1. The value r indicates the distance of the point from the z-axis. It is the radius of the cylinder.
2. The value Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis. It is basically measured from the x axis in the x-y plane. It is measured anti clockwise.
3. The value z indicates the distance of the point from z-axis. It is the same as in the Cartesian system. In short, it is the height of the cylinder.

**Range of the variables:**

It defines the minimum and the maximum values of r, Φ and z.

$0 \leq r \leq \infty$
$0 \leq \Phi \leq 2\pi$
$-\infty \leq z \leq \infty$

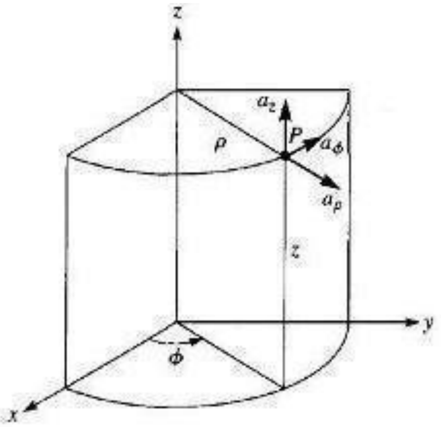



Figure shows Point P and Unit vectors in Cylindrical Co-ordinate System.

**Differential Displacement / Differential Length (dl):**

It is given as

$$\overline{dl} = dr\bar{a}_r + rd\varphi\bar{a}_\varphi + dz\bar{a}_z$$

Differential length for a line parallel to r, Φ and z axis are respectively given as:

dl = $dr\bar{a}_r$---( For a line parallel to r-direction).
dl = $rd\varphi\bar{a}_\varphi$ ---( For a line Parallel to Φ-direction).
dl = $dz\bar{a}_z$ ---( For a line parallel to z-axis).

**Differential Normal Surface (ds):**

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.

This surface describes a circular disc. Always remember- To define a circular disk we need two parameter one distance measure and one angular measure. An angular parameter will always give a curved line or an arc.

In this case dΦ is measured in terms of change in arc.

Arc is given as:
Arc= radius * angle

$$\overline{ds} = rdrd\varphi\bar{a}_z$$
$$\overline{ds} = drdz\bar{a}_\varphi$$
$$\overline{ds} = rdrd\varphi\bar{a}_r$$

**Differential Volume:**

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = rdrd\varphi dz$$

3. **Spherical coordinate System**:

Spherical coordinates consist of one scalar value (r), with units of distance, while the other two scalarvalues (θ, Φ) have angular units (degrees or radians).

A Vector in Spherical System is represented as ($A_r$ ,$A_\Theta$, $A_\Phi$) or

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\varphi\bar{a}_\varphi$$

Where$\bar{a}_r$,$\bar{a}_\theta$ and $\bar{a}_\varphi$ are the unit vectors in r, θ and Φ direction respectively.

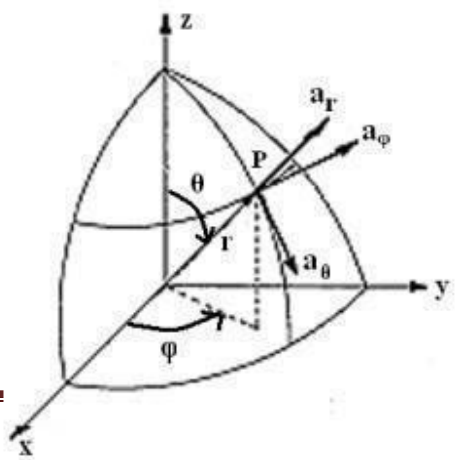
The physical significance of each parameter of spherical coordinates:

1. The value r expresses the distance of the point from origin (i.e. similar to altitude). It is the radius of the sphere.
2. The angle θ is the angle formed with the z- axis (i.e. similar to latitude). It is also called the co-latitude angle. It is measured clockwise.
3. The angle Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis (i.e. similar to longitude). It is basically measured from the x axis in the x-y plane. It is measured counter-clockwise.

**Range of the variables:**

It defines the minimum and the maximum value that r, θ and υ can have in spherical co-ordinate system.

$0 \le r \le \infty$
$0 \le \theta \le \pi$
$0 \le \Phi \le 2\pi$



**Differential length**:
It is given as
$\overline{dl} = dr\bar{a}_r + rd\theta\bar{a}_\theta + r\sin\theta\, d\varphi\bar{a}_\varphi$

Differential length for a line parallel to r, θ and Φ axis are respectively given as:

dl = $dr\bar{a}_r$--(For a line parallel to r axis)

dl = $rd\theta\bar{a}_\theta$---( For a line parallel to θ direction)

dl = $r\sin\theta\, d\varphi\bar{a}_\varphi$ --(For a line parallel to Φ direction)

**Differential Normal Surface (ds):**

Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.

$$\overline{ds} = rdrd\theta\bar{a}_\varphi$$
$$\overline{ds} = r^2\sin\theta\, d\varphi d\theta\bar{a}_r$$
$$\overline{ds} = r\sin\theta\, drd\varphi\bar{a}_\theta$$

**Differential Volume:**

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = r^2\sin\theta\, drd\varphi d\theta$$

## Coordinate transformations:

### Coordinate transformations

| Transformation | Coordinate Variables | Unit Vectors | Vector Components |
|---|---|---|---|
| Cartesian to cylindrical | $r = \sqrt[+]{x^2+y^2}$<br>$\phi = \tan^{-1}(y/x)$<br>$z = z$ | $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$<br>$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{z}}$ | $A_r = A_x\cos\phi + A_y\sin\phi$<br>$A_\phi = -A_x\sin\phi + A_y\cos\phi$<br>$A_z = A_z$ |
| Cylindrical to Cartesian | $x = r\cos\phi$<br>$y = r\sin\phi$<br>$z = z$ | $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$<br>$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{z}}$ | $A_x = A_r\cos\phi - A_\phi\sin\phi$<br>$A_y = A_r\sin\phi + A_\phi\cos\phi$<br>$A_z = A_z$ |
| Cartesian to spherical | $R = \sqrt[+]{x^2+y^2+z^2}$<br>$\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$<br>$\phi = \tan^{-1}(y/x)$ | $\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$<br>$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$<br>$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ | $A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$<br>$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$<br>$A_\phi = -A_x\sin\phi + A_y\cos\phi$ |
| Spherical to Cartesian | $x = R\sin\theta\cos\phi$<br>$y = R\sin\theta\sin\phi$<br>$z = R\cos\theta$ | $\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$<br>$\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ | $A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$<br>$A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$<br>$A_z = A_R\cos\theta - A_\theta\sin\theta$ |
| Cylindrical to spherical | $R = \sqrt[+]{r^2+z^2}$<br>$\theta = \tan^{-1}(r/z)$<br>$\phi = \phi$ | $\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$<br>$\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$<br>$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ | $A_R = A_r\sin\theta + A_z\cos\theta$<br>$A_\theta = A_r\cos\theta - A_z\sin\theta$<br>$A_\phi = A_\phi$ |
| Spherical to cylindrical | $r = R\sin\theta$<br>$\phi = \phi$<br>$z = R\cos\theta$ | $\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$<br>$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$<br>$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ | $A_r = A_R\sin\theta + A_\theta\cos\theta$<br>$A_\phi = A_\phi$<br>$A_z = A_R\cos\theta - A_\theta\sin\theta$ |

Vector relations in the three common coordinate systems.

| | Cartesian Coordinates | Cylindrical Coordinates | Spherical Coordinates |
|---|---|---|---|
| Coordinate variables | $x, y, z$ | $r, \phi, z$ | $R, \theta, \phi$ |
| Vector representation $\mathbf{A} =$ | $\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ | $\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ | $\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$ |
| Magnitude of A $\quad |\mathbf{A}| =$ | $\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$ | $\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$ | $\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$ |
| Position vector $\quad \overrightarrow{OP_1} =$ | $\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$ | $\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$ | $\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$ |
| Base vectors properties | $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$<br>$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$<br>$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$<br>$\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$<br>$\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$ | $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$<br>$\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$<br>$\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$<br>$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$<br>$\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$ | $\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$<br>$\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$<br>$\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$<br>$\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$<br>$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$ |
| Dot product $\quad \mathbf{A}\cdot\mathbf{B} =$ | $A_xB_x + A_yB_y + A_zB_z$ | $A_rB_r + A_\phi B_\phi + A_zB_z$ | $A_RB_R + A_\theta B_\theta + A_\phi B_\phi$ |
| Cross product $\quad \mathbf{A}\times\mathbf{B} =$ | $\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ | $\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$ | $\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$ |
| Differential length $\quad d\mathbf{l} =$ | $\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$ | $\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$ | $\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$ |
| Differential surface areas | $d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$<br>$d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$<br>$d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$ | $d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$<br>$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$<br>$d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$ | $d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$<br>$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$<br>$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$ |
| Differential volume $\quad dV =$ | $dx\,dy\,dz$ | $r\,dr\,d\phi\,dz$ | $R^2\sin\theta\,dR\,d\theta\,d\phi$ |

**DIVERGENCE THEOREM**:

It states that the net outward flux of a vector field A through a closed surface S is equal to the volume integral of the divergence of the field A inside the surface.

**STOKES THEOREM:**

It states that the circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L.

**Electrostatics:**

Electrostatics is a branch of science that involves the study of various phenomena caused by electric charges that are slow-moving or even stationary. Electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics as the study of electric charges at rest.

The two important laws of electrostatics are

- Coulomb‘s Law.
- Gauss‘s Law.

Both these laws are used to find the electric field due to different charge configurations.

Coulomb‘s law is applicable in finding electric field due to any charge configurations where as Gauss‘s law is applicable only when the charge distribution is symmetrical.

**Coulomb's Law**

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

A point charge is a charge that occupies a region of space which is negligibly small compared to the distance between the point charge and any other object.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

Mathematically, $F = \frac{kQ_1Q_2}{R^2}$ , where k is the proportionality constant.

In SI units, Q1 and Q2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$ , $\varepsilon_0$ is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where $\varepsilon_r$ is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ ....................... (1)

As shown in the Figure 1 let the position vectors of the point charges Q1and Q2 are given by $\vec{r_1}$ and $\vec{r_2}$ . Let $\overrightarrow{F_{12}}$ represent the force on Q1 due to charge Q2.

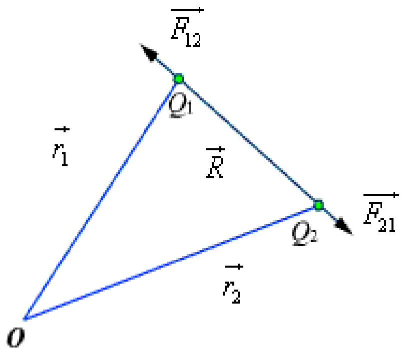



**Fig 1: Coulomb's Law**

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$\widehat{a_{12}} = \frac{\left(\vec{r_2} - \vec{r_1}\right)}{R}$ and $\widehat{a_{21}} = \frac{\left(\vec{r_1} - \vec{r_2}\right)}{R}$

$\overrightarrow{F_{12}}$ can be defined as $\overrightarrow{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{\left(\vec{r_2} - \vec{r_1}\right)}{\left|\vec{r_2} - \vec{r_1}\right|^3}$ .

Similarly the force on $Q_1$ due to charge $Q_2$ can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have $N$ number of charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located respectively at the points represented by the position vectors $\vec{r_1}, \vec{r_2}, \ldots\ldots \vec{r_N}$ , the force experienced by a charge $Q$ located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i\left(\vec{r} - \vec{r_i}\right)}{\left|\vec{r} - \vec{r_i}\right|^3}$$

**Electric Field:**

Electric field due to a charge is the space around the unit charge in which it experiences a force. Electric field intensity or the electric field strength at a point is defined as the force per unit charge.

Mathematically,

E = F / Q

OR

F = E Q

The force on charge Q is the product of a charge (which is a scalar) and the value of the electric field (which is a vector) at the point where the charge is located. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q}$$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r'}$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3}$$

For a collection of *N* point charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located at $\vec{r_1}, \vec{r_2}, \ldots\ldots \vec{r_N}$, the electric field intensity at point $\vec{r'}$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r_i})}{\left|\vec{r} - \vec{r_i}\right|^3}$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge *(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r'})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3} = \frac{\rho(\vec{r'})dv'(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r'}\right|^3}$$

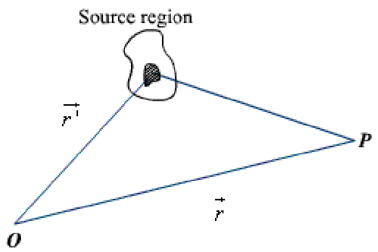



**Fig 2: Continuous Volume Distribution of Charge**

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(\vec{r})} = \int_V \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} dv'$$ ..............volume charge..........................

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(\vec{r})} = \int_L \frac{\rho_L(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} dl'$$ ....................line charge ...............

$$\overline{E(\vec{r})} = \int_S \frac{\rho_s(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} ds'$$ .................surface charge.....................

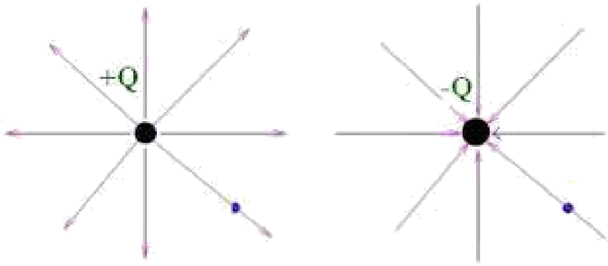
**Electric Lines of Forces:**

Electric line of force is a pictorial representation of the electric field.

Electric line of force (also called Electric Flux lines or Streamlines) is an imaginary straight or curved path along which a unit positive charge tends to move in an electric field.

**Properties Of Electric Lines Of Force:**

1. Lines of force start from positive charge and terminate either at negative charge or move to infinity.
2. Similarly lines of force due to a negative charge are assumed to start at infinity and terminate at the negative charge.




3. The number of lines per unit area, through a plane at right angles to the lines, is proportional to the magnitude of E. This means that, where the lines of force are close together, E is large and where they are far apart E is small.

4. If there is no charge in a volume, then each field line which enters it must also leave it.

5. If there is a positive charge in a volume then more field lines leave it than enter it.

6. If there is a negative charge in a volume then more field lines enter it than leave it.

7. Hence we say Positive charges are sources and Negative charges are sinks of the field.

8. These lines are independent on medium.

9. Lines of force never intersect i.e. they do not cross each other.

10. Tangent to a line of force at any point gives the direction of the electric field E at that point.

**Electricfluxdensity:**

As stated earlier electric field intensity or simply ‘Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon \vec{E}$$

We define the electric flux as

$$\psi = \int_S \vec{D} \cdot d\vec{s}$$

**Gauss's Law:**

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

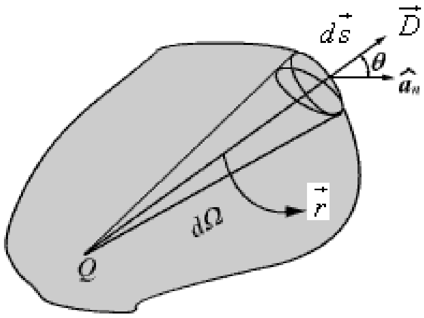



**Fig 3: Gauss's Law**

Let us consider a point charge *Q* located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance *r* on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2}ds\cos\theta$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*.

Therefore we can write $d\psi = \frac{Q}{4\pi}d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_s d\psi = \frac{Q}{4\pi}\oint_s d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Hence we have,

$$Q_{enc} = \oint_S D \,.\, ds = \int_V \rho_V \, dv$$

Applying Divergence theorem we have,

$$\oint_S D \,.\, ds = \int_V \nabla \cdot D \, dv$$

Comparing the above two equations, we have

$$\int_V \nabla \cdot D \, dv = \int_V \rho_V \, dv$$

This equation is called the 1st Maxwell's equation of electrostatics.

**Application of Gauss's Law:**

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. **$\vec{E}$ due to an infinite line charge**

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $_L$C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

Considering the fact that the unit normal vector to areas $S_1$ and $S_3$ are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write, $\rho_l l = \varepsilon_0 E.2\pi\rho l$

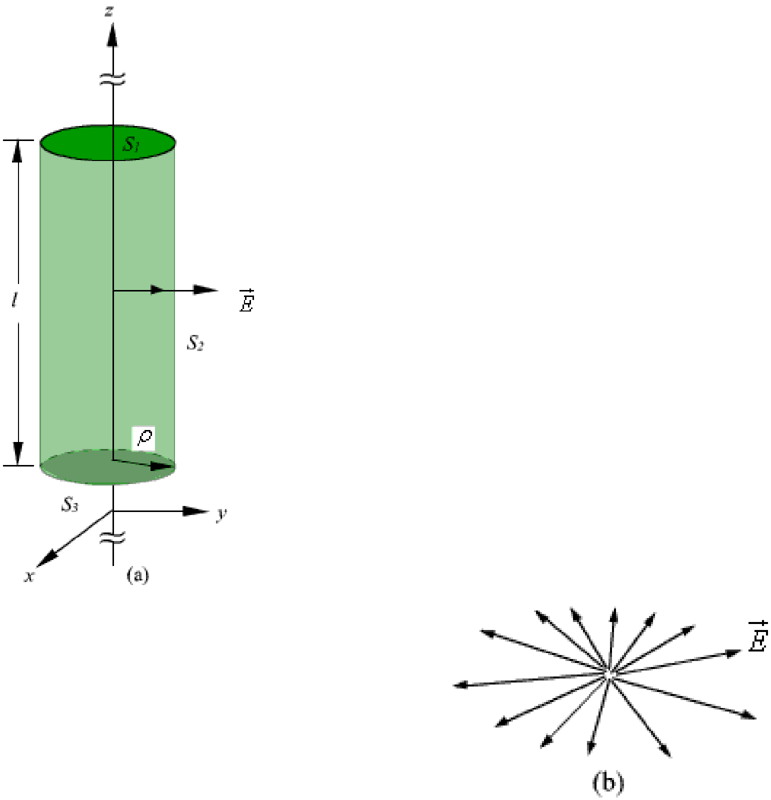



**Fig 4: Infinite Line Charge**

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

**2. Infinite Sheet of Charge**

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x*-*z* plane as shown in figure 5. Assuming a surface charge density of $\rho_s$ for the infinite surface charge, if we consider a cylindrical volume having sides $\Delta s$ placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D} \cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \qquad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

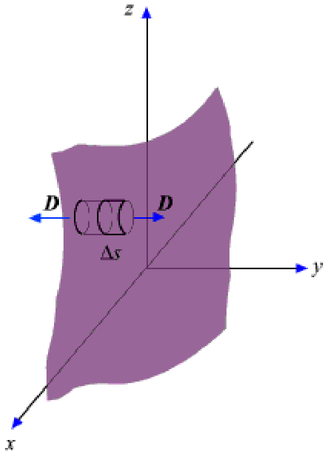



**Fig 5: Infinite Sheet of Charge**

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

**3. Uniformly Charged Sphere**

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \le r_0$ ; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3} \pi r^3$$

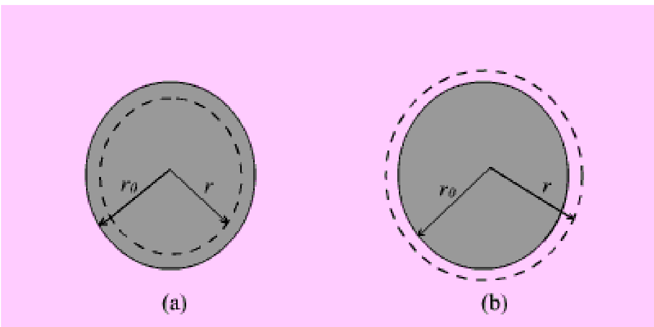



**Fig 6: Uniformly Charged Sphere**

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$

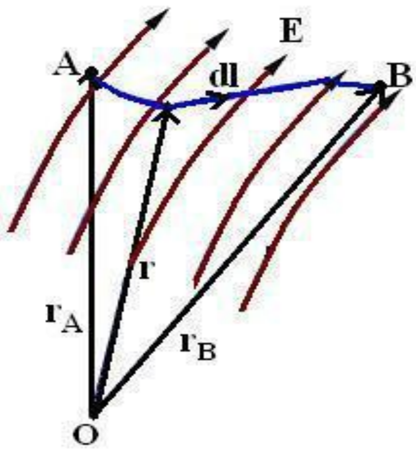
**Electric Potential / Electrostatic Potential (V):**

If a charge is placed in the vicinity of another charge (or in the field of another charge), it experiences a force. If a field being acted on by a force is moved from one point to another, then work is either said to be done on the system or by the system.

Say a point charge Q is moved from point A to point B in an electric field E, then the work done in moving the point charge is given as:

$W_{A\to B} = -\int_A^B (F \cdot dl) = -Q\int_A^B (E \cdot dl)$

where the – ve sign indicates that the work is done on the system by an external agent.



The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(VAB).

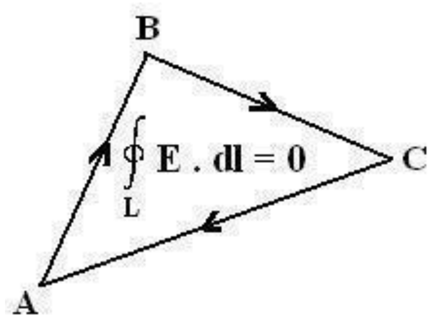
VAB = WA→B / Q

- ∫AB(E . dl)

- ∫InitialFinal (E . dl)

If the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

The electrostatic field is conservative i.e. the value of the line integral depends only on end points and is independent of the path taken.




- Since the electrostatic field is conservative, the electric potential can also be written as:

$$V_{AB} = -\int_A^B \bar{E}.\bar{dl}$$

$$V_{AB} = -\int_A^{p_0} \bar{E}.\bar{dl} - \int_{p_0}^B \bar{E}.\bar{dl}$$

$$V_{AB} = -\int_{p_0}^B \bar{E}.\bar{dl} + \int_{p_0}^A \bar{E}.\bar{dl}$$

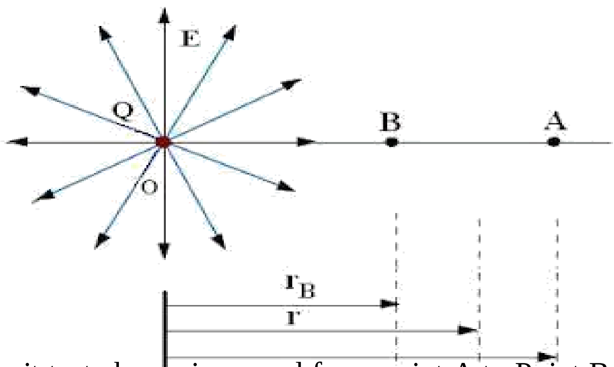
$$V_{AB} = V_B - V_A$$

Thus the potential difference between two points in an electrostatic field is a scalar field that is defined at every point in space and is independent of the path taken.

- The work done in moving a point charge from point A to point B can be written as:

$W_{A \rightarrow B} = - Q\,[V_B - V_A] = -Q\int_A^B \bar{E}\,.\,\bar{dl}$

- Consider a point charge Q at origin O.




Now if a unit test charge is moved from point A to Point B, then the potential difference between them is given as:

$$V_{AB} = -\int_A^B \mathbf{E}\,.\,\mathbf{dl} = -\int_{r_A}^{r_B} \mathbf{E}\,.\,\mathbf{dl} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a_r}\cdot dr\,\mathbf{a_r}$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = V_B - V_A$$

- Electrostatic potential or Scalar Electric potential (V) at any point P is given by:

$$V = -\int_{P_0}^{P} \bar{E}\,.\,\bar{dl}$$

The reference point Po is where the potential is zero (analogues to ground in a circuit). The reference is often taken to be at infinity so that the potential of a point in space is defined as

$$V = -\int_{\infty}^{P} \bar{E}\,.\bar{dl}$$

Basically potential is considered to be zero at infinity. Thus potential at any point ( rB = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge frominfinity to that point (i.e. rA → ∞)

Electric potential (V) at point r due to a point charge Q located at a point with position vector r1 is given as:

$$V = \frac{Q}{4\pi\varepsilon|r - r_1|}$$

Similarly for N point charges Q1, Q2 ….Qn located at points with position vectors r1, r2, r3…..rn, theelectric potential (V) at point r is given as:

$$V = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{Q_k}{|r - r_k|} \qquad V = \frac{Q}{4\pi\varepsilon r}$$

The charge element dQ and the total charge due to different charge distribution is given as:

dQ = ρldl → Q = ∫L (ρldl) → (Line Charge)

dQ = ρsds → Q = ∫S (ρsds) → (Surface Charge)

dQ = ρvdv → Q = ∫V (ρvdv) → (Volume Charge)

$$V = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon\,|r - r_1|} \quad \textbf{(Line Charge)}$$

$$V = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon\,|r - r_1|} \quad \textbf{(Surface Charge)}$$

$$V = \int_V \frac{\rho_V\, dv}{4\pi\varepsilon\,|r - r_1|} \quad \textbf{(Volume Charge)}$$

## Second Maxwell's Equation of Electrostatics:

The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points($V_{AB}$).

$V_{AB} = V_B – V_A$

Similarly,

$V_{BA} = V_A – V_B$

Hence it‗s clear that potential difference is independent of the path taken. Therefore

$V_{AB} = - V_{BA}$

$V_{AB} + V_{BA} = 0$

∫AB (E . dl) + [ - ∫BA (E . dl) ] = 0

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = 0$$

The above equation is called the second Maxwell‗s Equation of Electrostatics in integral form..
The above equation shows that the line integral of Electric field intensity (E) along a closed path is equal to zero.
In simple words—No work is done in moving a charge along a closed path in an electrostatic field.

Applying Stokes‗ Theorem to the above Equation, we have:

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{ds} = 0$$

$$\text{--->}\ \nabla \times \mathbf{E} = 0$$

If the Curl of any vector field is equal to zero, then such a vector field is called an Irrotational or Conservative Field. Hence an electrostatic field is also called a conservative field.
The above equation is called the second Maxwell‗s Equation of Electrostatics in differential form.

## Relationship Between Electric Field Intensity (E) and Electric Potential (V):

Since Electric potential is a scalar quantity, hence dV (as a function of x, y and z variables) can be written as:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right)\cdot\left(dx\,a_x + dy\,a_y + dz\,a_z\right) = -E.dl$$

$$\nabla V.dl = -E.dl \quad \text{--->} \quad \boxed{E = -\nabla V}$$

Hence the Electric field intensity (E) is the negative gradient of Electric potential (V).
The negative sign shows that E is directed from higher to lower values of V i.e. E is opposite to the direction in which V increases.

## Energy Density In Electrostatic Field / Work Done To Assemble Charges:

In case, if we wish to assemble a number of charges in an empty system, work is required to do so. Also electrostatic energy is said to be stored in such a collection.

Let us build up a system in which we position three point charges Q1, Q2 and Q3 at position r1, r2 and r3 respectively in an initially empty system.

Consider a point charge Q1 transferred from infinity to position r1 in the system. It takes no work to bring the first charge from infinity since there is no electric field to fight against (as the system is empty i.e. charge free).

Hence, W1 = 0 J

Now bring in another point charge Q2 from infinity to position r2 in the system. In this case we have to do work against the electric field generated by the first charge Q1.

Hence, W2 = Q2 V21

where V21 is the electrostatic potential at point r2 due to Q1.

- Work done W2 is also given as:

$$W_2 = \frac{Q_2 Q_1}{4\pi\varepsilon\,|r_2 - r_1|}$$

Now bring in another point charge Q3 from infinity to position r3 in the system. In this case we have to do work against the electric field generated by Q1 and Q2.

Hence, W3 = Q3 V31 + Q3 V32 = Q3 ( V31 + V32 )

where V31 and V32 are electrostatic potential at point r3 due to Q1 and Q2 respectively.

The work done is simply the sum of the work done against the electric field generated by point charge Q1 and Q2 taken in isolation:

$$W_3 = \frac{Q_3 Q_1}{4\pi\varepsilon\,|r_3 - r_1|} + \frac{Q_3 Q_2}{4\pi\varepsilon\,|r_3 - r_2|}$$

- Thus the total work done in assembling the three charges is given as:

WE = W1 + W2 + W3

0 + Q2 V21 + Q3 ( V31 + V32 )

Also total work done ( WE ) is given as:

$$W_E = \frac{1}{4\pi\varepsilon}\left(\frac{Q_2 Q_1}{|r_2 - r_1|} + \frac{Q_3 Q_1}{|r_3 - r_1|} + \frac{Q_3 Q_2}{|r_3 - r_2|}\right)$$

If the charges were positioned in reverse order, then the total work done in assembling them is given as:

WE = W3 + W2+ W1
= 0 + Q2V23 + Q3( V12+ V13)

Where V23 is the electrostatic potential at point r2 due to Q3 and V12 and V13 are electrostatic potential at point r1 due to Q2 and Q3 respectively.

- Adding the above two equations we have,

2WE = Q1 (V12 + V13) + Q2 (V21 + V23) + Q3 (V31 + V32)

= Q1 V1 + Q2 V2 + Q3 V3

Hence

WE =1 / 2 [Q1V1 + Q2V2 + Q3V3]

where V1, V2 and V3 are total potentials at position r1, r2 and r3 respectively.

- The result can be generalized for N point charges as:

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

The above equation has three interpretation: This equation represents the potential energy of the system.This is the work done in bringing the static charges from infinity and assembling them in the required system. This is the kinetic energy which would be released if the system gets dissolved i.e. the charges returns back to infinity.

In place of point charge, if the system has continuous charge distribution ( line, surface or volume charge), then the total work done in assembling them is given as:

$$W_E = \frac{1}{2}\int_L \rho_L V\, dl \quad \text{(Line Charge)}$$

$$W_E = \frac{1}{2}\int_S \rho_S V\, ds \quad \text{(Surface Charge)}$$

$$W_E = \frac{1}{2}\int_V \rho_V V\, dv \quad \text{(Volume Charge)}$$

Since $\rho v = \nabla . D$ and $E = - \nabla V$,

Substituting the values in the above equation, work done in assembling a volume charge distribution in terms of electric field and flux density is given as:

$$W_E = \frac{1}{2}\int_V D . E \, dv = \frac{1}{2}\int_V \varepsilon E^2 \, dv$$

The above equation tells us that the potential energy of a continuous charge distribution is stored in an electric field.

The electrostatic energy density wE is defined as:

$$w_E = \frac{1}{2}\varepsilon E^2 \quad ; \quad W_E = \int_V w_E \, dv$$

**ELECTROSTATICS-II**

**Properties of Materials and Steady Electric Current:**

Electric field can not only exist in free space and vacuum but also in any material medium. When an electric field is applied to the material, the material will modify the electric field either by strengthening it or weakening it, depending on what kind of material it is.

Materials are classified into 3 groups based on conductivity / electrical property:

- Conductors (Metals like Copper, Aluminum, etc.) have high conductivity ($\sigma >> 1$).
- Insulators / Dielectric (Vacuum, Glass, Rubber, etc.) have low conductivity ($\sigma << 1$).
- Semiconductors (Silicon, Germanium, etc.) have intermediate conductivity.

Conductivity ($\sigma$) is a measure of the ability of the material to conduct electricity. It is the reciprocal of resistivity ($\rho$). Units of conductivity are Siemens/meter and mho.

The basic difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current. Conductors have a large amount of free electrons where as insulators have only a few number ofelectrons for conduction of current. Most of the conductors obey ohm‘s law. Such conductors are also called ohmic conductors.

Due to the movement of free charges, several types of electric current can be caused. The different types of electric current are:

- Conduction Current.
- Convection Current.
- Displacement Current.

**Electric current:**

Electric current (I) defines the rate at which the net charge passes through a wire of cross sectional surface area S.

Mathematically,

If a net charge ΔQ moves across surface S in some small amount of time Δt, electric current(I) is defined as:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

How fast or how speed the charges will move depends on the nature of the material medium.

**Current density:**

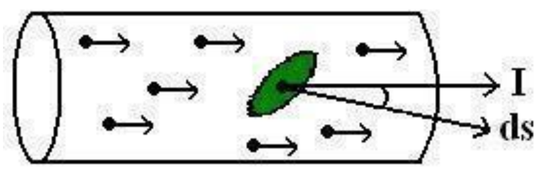
Current density (J) is defined as current ΔI flowing through surface ΔS.

Imagine surface area ΔS inside a conductor at right angles to the flow of current. As the area approaches zero, the current density at a point is defined as:

$$J = \lim_{\Delta s \to 0} \frac{\Delta I}{\Delta S}$$

The above equation is applicable only when current density (J) is normal to the surface.

In case if current density(J) is not perpendicular to the surface, consider a small area ds of the conductor at an angle θ to the flow of current as shown:




In this case current flowing through the area is given as:

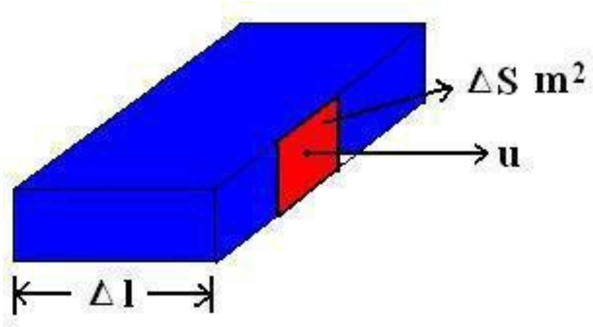
dI = J dS cosθ = J . dS and $I = \int_S \bar{J} . \overline{ds}$

Where angle θ is the angle between the normal to the area and direction of the current.

From the above equation it‘s clear that electric current is a scalar quantity.

**CONVECTION CURRENT DENSITY**:

Convection current occurs in insulators or dielectrics such as liquid, vacuum and rarified gas. Convection current results from motion of electrons or ions in an insulating medium. Since convection current doesn‘t involve conductors, hence it does not satisfy ohm‘s law. Consider a filament where there is a flow of charge ρv at a velocity u = uy ay.




- Hence the current is given as:

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

But we know $\boxed{\Delta Q = \rho_V \Delta V}$

Hence

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_V \Delta V}{\Delta t} = \rho_V \Delta S \frac{\Delta l}{\Delta t}$$

$$= \rho_V \Delta S \, u_y$$

Again, we also know that $\boxed{J_y = \frac{\Delta I}{\Delta S}}$

Hence $$J_y = \frac{\Delta I}{\Delta S} = \rho_V u_y$$

Where uy is the velocity of the moving electron or ion and $\rho_v$ is the free volume charge density.

- Hence the convection current density in general is given as:

$$J = \rho_v u$$

**Conduction Current Density:**

Conduction current occurs in conductors where there are a large number of free electrons. Conduction current occurs due to the drift motion of electrons (charge carriers). Conduction current obeys ohm‘s law.

When an external electric field is applied to a metallic conductor, conduction current occurs due to the drift of electrons.

The charge inside the conductor experiences a force due to the electric field and hence should accelerate but due to continuous collision with atomic lattice, their velocity is reduced. The net effect is that the electrons moves or drifts with an average velocity called the drift velocity ($\upsilon d$) which is proportional to the applied electric field (E).

Hence according to Newton‘s law, if an electron with a mass m is moving in an electric field E with anaverage drift velocity $\upsilon d$, the the average change in momentum of the free electron must be equal to the applied force (F = - e E).

$$\frac{m\upsilon_d}{\tau} = -eE$$

**where $\tau$ is the average time interval between collision.**

$$\upsilon_d = \left(-\frac{e\tau}{m}\right)E$$

The drift velocity per unit applied electric field is called the mobility of electrons ($\mu e$).

$\upsilon d = -\mu e\, E$

where $\mu e$ is defined as:

$$\mu_e = \left(-\frac{e\tau}{m}\right)$$

Consider a conducting wire in which charges subjected to an electric field are moving with drift velocity $\upsilon_d$.
Say there are Ne free electrons per cubic meter of conductor, then the free volume charge density($\rho_v$)within the wire is

$\rho_v = -\,e\,N_e$

The charge $\Delta Q$ is given as:

$\Delta Q = \rho_v\,\Delta V = -\,e\,N_e\,\Delta S\,\Delta l = -\,e\,N_e\,\Delta S\,\upsilon_d\,\Delta t$

- The incremental current is thus given as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = -N_e\,e\,\Delta S\,\upsilon_d$$

**Now since** $\boxed{\upsilon_d = -\,\mu_e\,E}$

**Therefore**

$$\Delta I = N_e\,e\,\Delta S\,\mu_e E$$

The conduction current density is thus defined as:

$$J_c = \frac{\Delta I}{\Delta S} = N_e\,e\,\mu_e E = \sigma E$$

where σ is the conductivity of the material.

The above equation is known as the Ohm's law in point form and is valid at every point in space.

In a semiconductor, current flow is due to the movement of both electrons and holes, hence conductivity is given as:

$\sigma = (N_e\,\mu_e + N_h\,\mu_h)e$

**DIELECTRC CONSTANT:**

It is also known as Relative permittivity.

If two charges q 1 and q 2 are separated from each other by a small distance r. Then by using the coulombs law of forces the equation formed will be

$$\mathbf{F}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

In the above equation $\varepsilon_0$ is the electrical permittivity or you can say it, Dielectric constant.

If we repeat the above case with only one change i.e. only change in the separation medium between the charges. Here some material medium must be used. Then the equation formed will be.

$$\mathbf{F}_m = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

Now after division of above two equations

$$\frac{\mathbf{F}_0}{\mathbf{F}_m} = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_r \text{ Or } k$$

In the above figure

$\varepsilon_\mathbf{r}$ is the Relative Permittivity. Again one thing to notice is that the dielectric constant is represented by the symbol (K) but permittivity by the symbol $\varepsilon_\mathbf{r}$

**CONTINUITY EQUATION:**

The continuity equation is derived from two of Maxwell's equations. It states that the divergence of the current density is equal to the negative rate of change of the charge density,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Derivation

One of Maxwell's equations, Ampère's law, states that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking the divergence of both sides results in

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t},$$

but the divergence of a curl is zero, so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = 0. \qquad (1)$$

Another one of Maxwell's equations, Gauss's law, states that

$$\nabla \cdot \mathbf{D} = \rho.$$

Substitute this into equation (1) to obtain

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation.

## 1.13 RELAXATION TIME:

- Let us consider that a charge is introduced at some interior point of a given material (conductor or dielectric).
- From, continuity of current equation, we have

$$\overline{J} = \frac{-rf_v}{rt} ----(1)$$

- We have, the point form of Ohm's law as,

$$\overline{J} = 6\overline{E} --- (2)$$

- From Gauss's law, we have,

$$\nabla\overline{D} = f_v \Rightarrow \in \nabla.\overline{E} = f_v \left[\because \overline{D} = \in .\overline{E}\right]$$

$$\therefore \nabla.\overline{E} = \frac{f_v}{\in} ------(1)$$

- Substitute equations (2) and (3) in equation (1), we get

$$\nabla.6\overline{E}f = 6.\nabla.\overline{E} = 6.\frac{f_v}{\in} = \frac{-\partial f_v}{\partial t}$$

$$\Rightarrow \frac{\partial f_v}{\partial t} + \frac{6}{\in}.f_v = 0 ------(4)$$

- The above equation is a homogeneous linear ordinary differential equation. By separating variable in eq (4), we get,

$$\frac{\partial f_v}{\partial t} = \frac{-6}{\in}.f_v$$

$$\Rightarrow \frac{\partial f_v}{\partial t} = \frac{-6}{\in}.\partial t$$

- Now integrate on both sides of above equation

$$\int \frac{\partial f_v}{\partial t} = -\frac{6}{\in}.\int \partial t$$

$$\Rightarrow \ln f_v = -\frac{6}{\in}.t + \ln f_{v0}$$

Where ln $p_{v0}$ is a constant of integration.

Thus,

$$\boxed{f_v = f_{v0} e^{-t/Tr}} \quad -----(5)$$

$$\boxed{T_r = \frac{\in}{6}}$$

- In eq (5), $f_{v0}$ is the initial charge density (i.e fv at t=0).
- We can see from the equation that as a result of introducing charge at some interior point of the material there is a decay of volume charge density $f_v$.
- The time constant "$T_r$" is known as the relaxation time or rearrangement time.
- Relaxation time is the time it takes a charge placed in the interior of a material to drop to $e^{-1}$ = 36.8 percent f its initial value.
- The relation time is short for good conductors and long for good dielectrics.

## LAPLACE'S AND POISSON'S EQUATIONS:

A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it. The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$E$ = electric field
$\rho$ = charge density
$\varepsilon_0$ = permittivity

and the electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes LaPlace's equation

$$\nabla^2 V = 0$$

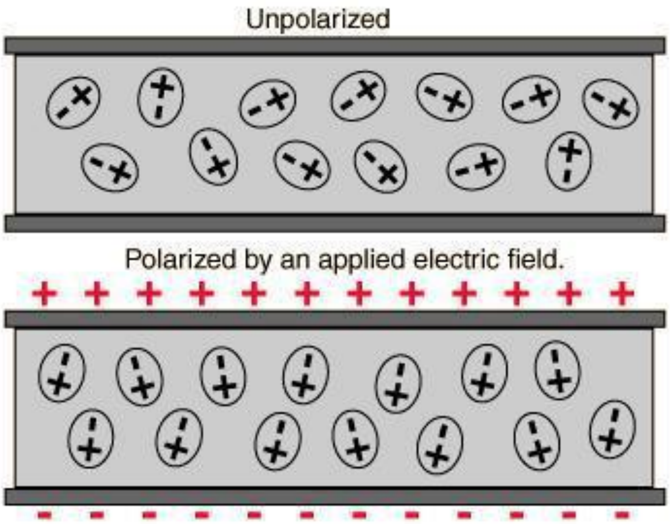
This mathematical operation, the divergence of the gradient of a function, is called the LaPlacian. Expressing the LaPlacian in different coordinate systems to take advantage of the symmetry of a charge distribution helps in the solution for the electric potential V. For example, if the charge distribution has spherical symmetry, you use the LaPlacian in spherical polar coordinates.

Since the potential is a scalar function, this approach has advantages over trying to calculate the electric field directly. Once the potential has been calculated, the electric field can be computed by taking the gradient of the potential.

**Polarization of Dielectric:**

If a material contains polar molecules, they will generally be in random orientations when no electric field is applied. An applied electric field will polarize the material by orienting the dipole moments of polar molecules.

This decreases the effective electric field between the plates and will increase the capacitance of the parallel plate structure. The dielectric must be a good electric insulator so as to minimize any DC leakage current through a capacitor.




The presence of the dielectric decreases the electric field produced by a given charge density.

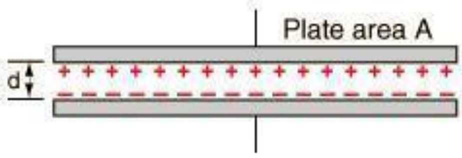
$$E_{effective} = E - E_{polarization} = \frac{\sigma}{k\varepsilon_0}$$

The factor k by which the effective field is decreased by the polarization of the dielectric is called the dielectric constant of the material.

## Capacitance:

The capacitance of a set of charged parallel plates is increased by the insertion of adielectric material. The capacitance is inversely proportional to the electric field between the plates, and the presence of the dielectric reduces the effective electric field. The dielectric is characterized by a dielectric constant k, and the capacitance is multiplied by that factor.

Parallel Plate Capacitor




$$C = \frac{\varepsilon A}{d} = \frac{k\varepsilon_0 A}{d}$$

Show

The capacitance of flat, parallel metallic plates of area A and separation d is given by the expression above where:

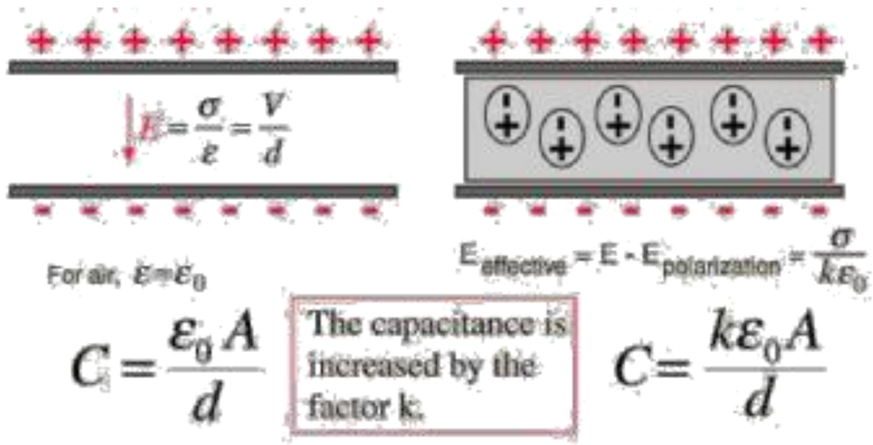
$$\varepsilon_0 = 8.854 \ x \ 10^{-12} \ \ F/m$$

= permittivity of space and

k = relative permittivity of the dielectric material between the plates.

k=1 for free space, k>1 for all media, approximately =1 for air.

The Farad, F, is the SI unit for capacitance and from the definition of capacitance is seen to be equal to a Coulomb/Volt.




**Series and parallel Connection of capacitors**

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 1. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2} \quad .......................(1)$$

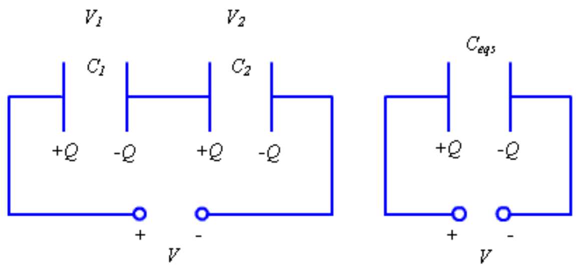



Fig 1.: Series Connection of Capacitors

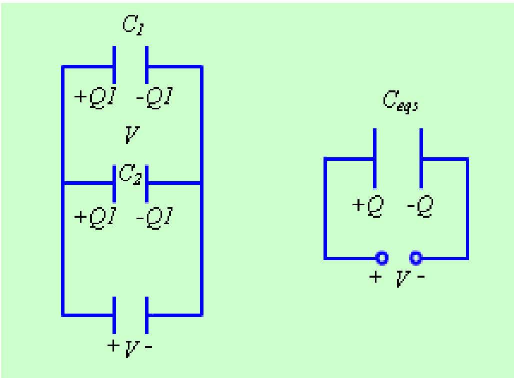



Fig 2: Parallel Connection of Capacitors

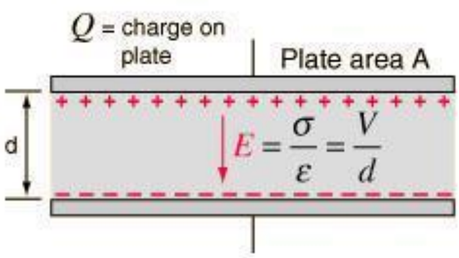
The same approach may be extended to more than two capacitors connected in series.

Parallel Case: For the parallel case, the voltages across the capacitors are the same.

The total charge $Q = Q_1 + Q_2 = C_1V + C_2V$

Therefore, $$C_{eqp} = \frac{Q}{V} = C_1 + C_2$$

## Capacitance of Parallel Plates:




The electric field between two large parallel plates is given by

$$E = \frac{\sigma}{\varepsilon} \quad \text{where} \quad \begin{array}{l} \sigma = \text{charge density} \\ \varepsilon = \text{permittivity} \end{array}$$

$$\text{and } \sigma = \frac{Q}{A}$$

The voltage difference between the two plates can be expressed in terms of the work done on a positive test charge q when it moves from the positive to the negative plate.

$$V = \frac{work\ \ done}{charge} = \frac{Fd}{q} = Ed$$

It then follows from the definition of capacitance that

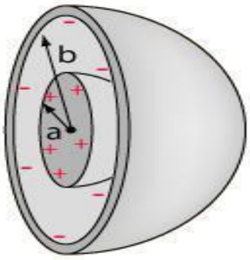
$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q\varepsilon}{\sigma d} = \frac{QA\varepsilon}{Qd} = \frac{A\varepsilon}{d}$$

## Spherical Capacitor:

The capacitance for spherical or cylindrical conductors can be obtained by evaluating
the voltage difference between the conductors for a given charge on each.

By applying Gauss' law to an charged conducting sphere, the electric field outside it is **found to be**

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$




The voltage between the spheres can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

From the definition of capacitance, the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

**Isolated Sphere Capacitor:**
An isolated charged conducting sphere has capacitance. Applications for such a capacitor may not be immediately evident, but it does illustrate that a charged sphere has stored some energy as a result of being charged. Taking the concentric sphere capacitance expression:

$$C = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$C = 4\pi\varepsilon_0 R$$

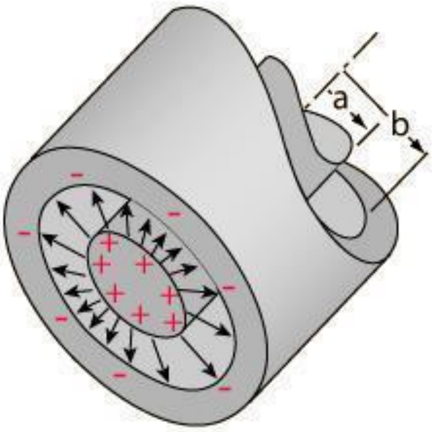
and taking the limits $a \to R$ and $b \to \infty$ gives

Further confirmation of this comes from examining the potential of a charged conducting sphere:

$$V = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{so at the surface } C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

**Cylindrical Capacitor:**

For a cylindrical geometry like a coaxial cable, the capacitance is usually stated as a capacitance per unit length. The charge resides on the outer surface of the inner conductor and the inner wall of the outer conductor. The capacitance expression is




$$\frac{C}{L} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

The capacitance for cylindrical orspherical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each. By applying Gauss' law to an infinite cylinder in a vacuum, the electric field outside a charged cylinder is found to be

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

The voltage between the cylinders can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{b}{a}\right] \qquad \frac{C}{L} = \frac{\lambda}{\Delta V} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

From the definition of capacitance and including the case where the volume is filled by a dielectric of dielectric constant k, the capacitance per unit length is defined above.

## Solved problems:

**Problem1:**

Find the charge in the volume defined by $0 \le x \le 1$ m, $0 \le y \le 1$ m, and $0 \le z \le 1$ m if $\rho = 30x^2y$ ($\mu$C/m$^3$). What change occurs for the limits $-1 \le y \le 0$ m?

Since $dQ = \rho\, dv$,

$$Q = \int_0^1\int_0^1\int_0^1 30x^2y\,dx\,dy\,dz = 5\ \mu\text{C}$$

For the change in limits on $y$,

$$Q = \int_0^1\int_{-1}^0\int_0^1 30x^2y\,dx\,dy\,dz = -5\ \mu\text{C}$$

**Problem-2**

Three point charges, $Q_1 = 30$ nC, $Q_2 = 150$ nC, and $Q_3 = -70$ nC, are enclosed by surface $S$. What net flux crosses $S$?

Since electric flux was defined as originating on positive charge and terminating on negative charge, part of the flux from the positive charges terminates on the negative charge.

$$\Psi_{\text{net}} = Q_{\text{net}} = 30 + 150 - 70 = 110\ \text{nC}$$

**Problem-3**

A point charge, $Q = 30$ nC, is located at the origin in cartesian coordinates. Find the electric flux density $\mathbf{D}$ at $(1, 3, -4)$ m.

Referring to Fig. 3.12,

$$\mathbf{D} = \frac{Q}{4\pi R^2}\mathbf{a}_R$$

$$= \frac{30\times10^{-9}}{4\pi(26)}\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)$$

$$= (9.18 \times 10^{-11})\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)\ \text{C/m}^2$$

or, more conveniently, $D = 91.8$ pC/m$^2$.

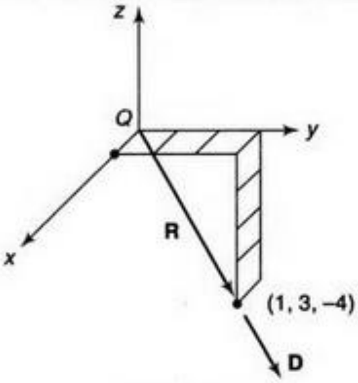



Fig. 3.12

**Problem-4**

Given that $\mathbf{D} = 10x\mathbf{a}_x$ (C/m$^2$), determine the flux crossing a 1-m$^2$ area that is normal to the $x$ axis at $x = 3$ m.

Since $\mathbf{D}$ is constant over the area and perpendicular to it,

$$\Psi = DA = (30\ \text{C/m}^2)(1\ \text{m}^2) = 30\ \text{C}$$

**Problem-5**

Given the vector field $\mathbf{A} = 5x^2\left(\sin\frac{\pi x}{2}\right)\mathbf{a}_x$, find div $\mathbf{A}$ at $x = 1$.

$$\text{div }\mathbf{A} = \frac{\partial}{\partial x}\left(5x^2\sin\frac{\pi x}{2}\right) = 5x^2\left(\cos\frac{\pi x}{2}\right)\frac{\pi}{2} + 10x\sin\frac{\pi x}{2} = \frac{5}{2}\pi x^2\cos\frac{\pi x}{2} + 10x\sin\frac{\pi x}{2}$$

and $\quad \text{div }\mathbf{A}\big|_{x=1} = 10.$

**Problem-6**

Given that $\mathbf{D} = (10r^3/4)\mathbf{a}_r$ (C/m$^2$) in the region $0 < r \le 3$ m in cylindrical coordinates and $\mathbf{D} = (810/4r)\mathbf{a}_r$ (C/m$^2$) elsewhere, find the charge density.

For $\quad 0 < r \le 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{10r^4}{4}\right) = 10r^2 \text{ (C/m}^3\text{)}$$

and for $\quad r > 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}(810/4) = 0$$

**Problem-7**

An electrostatic field is given by $\mathbf{E} = (x/2 + 2y)\mathbf{a}_x + 2x\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu$C (a) from the origin to (4, 0, 0) m, and (b) from (4, 0, 0) m to (4, 2, 0) m.

**(a)** The first path is along the $x$ axis, so that $d\mathbf{l} = dx\,\mathbf{a}_x$.

$$dW = -Q\mathbf{E}\cdot d\mathbf{l} = (20 \times 10^{-6})\left(\frac{x}{2} + 2y\right)dx$$

$$W = (20 \times 10^{-6})\int_0^4\left(\frac{x}{2} + 2y\right)dx = 80\ \mu\text{J}$$

**(b)** The second path is in the $\mathbf{a}_y$ direction, so that $d\mathbf{l} = dy\mathbf{a}_y$.

$$W = (20 \times 10^{-6})\int_0^2 2x\,dy = 320\ \mu\text{J}$$

**Problem-8**

What electric field intensity and current density correspond to a drift velocity of $6.0 \times 10^{-4}$ m/s in a silver conductor?

For silver $\quad \sigma = 61.7$ MS/m $\quad$ and $\quad \mu = 5.6 \times 10^{-3}$ m$^2$/V · s.

$$E = \frac{U}{\mu} = \frac{6.0 \times 10^{-4}}{5.6 \times 10^{-3}} = 1.07 \times 10^{-1} \text{ V/m}$$

$$J = \sigma E = 6.61 \times 10^6 \text{ A/m}^2$$

**Problem-9**

Find the current in the circular wire shown in Fig. 6.6 if the current density is $\mathbf{J} = 15(1 - e^{-1000r})\mathbf{a}_z$ (A/m$^2$). The radius of the wire is 2 mm.

A cross section of the wire is chosen for $S$. Then

$$dI = \mathbf{J} \cdot d\mathbf{S}$$
$$= 15(1 - e^{-1000r})\mathbf{a}_z \cdot r\,dr\,d\phi\,\mathbf{a}_z$$

and

$$I = \int_0^{2\pi}\int_0^{0.002} 15(1 - e^{-1000r})r\,dr\,d\phi$$
$$= 1.33 \times 10^{-4}\ \text{A} = 0.133\ \text{mA}$$

Any surface $S$ which has a perimeter that meets the outer surface of the conductor all the way around will have the same total current, $I = 0.133$ mA, crossing it.

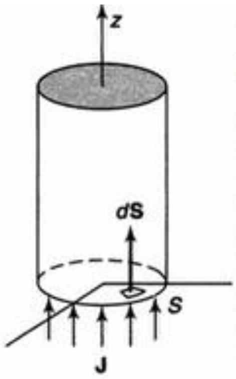



**Fig. 6.6**

**Problem-10**

Determine the relaxation time for silver, given that $\sigma = 6.17 \times 10^7$ S/m. If charge of density $\rho_0$ is placed within a silver block, find $\rho$ after one, and also after five, time constants.

Since $\varepsilon \approx \varepsilon_0$,

$$\tau = \frac{\varepsilon}{\sigma} = \frac{10^{-9}36\pi}{6.17 \times 10^7} = 1.43 \times 10^{-19}\ \text{s}$$

Therefore

at $t = \tau$: $\rho = \rho_0 e^{-1} = 0.368\rho_0$

at $t = 5\tau$: $\rho = \rho_0 e^{-5} = 6.74 \times 10^{-3}\rho_0$

**Problem-11**

Find the magnitudes of $\mathbf{D}$ and $\mathbf{P}$ for a dielectric material in which $E = 0.15$ MV/m and $\chi_e = 4.25$.

Since $\varepsilon_r = \chi_e + 1 = 5.25$,

$$D = \varepsilon_0\varepsilon_r E = \frac{10^{-9}}{36\pi}(5.25)(0.15 \times 10^6) = 6.96\ \mu\text{C/m}^2$$

$$P = \chi_e\varepsilon_0 E = \frac{10^{-9}}{36\pi}(4.25)(0.15 \times 10^6) = 5.64\ \mu\text{C/m}^2$$

**Problem-12**

In order to illustrate the application of (13) or (14), let us find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Fig. 2.4.

***Solution.*** We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Since $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\ \text{V} \cdot \text{m}$, we may now use (13) or (14) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z\ \text{V/m}$$

**Problem-13**

**Ex.** *A charge $Q_1 = -20\mu C$ is located at P (– 6, 4, 6) and a charge $Q_2 = 50\mu C$ is located at R (5, 8, – 2) in a free space. Find the force exerted on $Q_2$ by $Q_1$ in vector form. The distances given are in metres.*

**Sol. :** From the co-ordinates of P and R , the respective position vectors are –

$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$$

and

$$\bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$$

The force on $Q_2$ is given by,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = [5-(-6)]\ \bar{a}_x + (8-4)\ \bar{a}_y + [-2-(6)\bar{a}_z]$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$\therefore \quad |R_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2} = 14.1774$$

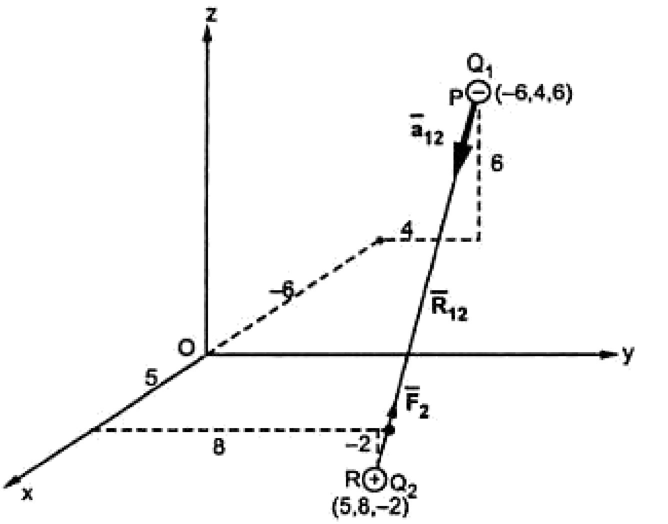



Fig. 2.5

$$\therefore \quad \bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\,\bar{a}_x + 4\,\bar{a}_y - 8\,\bar{a}_z}{14.1774}$$

$$\therefore \quad \bar{a}_{12} = 0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z$$

$$\therefore \quad \bar{F}_2 = \frac{-20\times 10^{-6}\times 50\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times (14.1774)^2}[\bar{a}_{12}]$$

$$= -0.0447\left[0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z\right] \quad \text{... (A)}$$

$$= -0.0346\,\bar{a}_x - 0.01261\,\bar{a}_y + 0.02522\,\bar{a}_z \text{ N} \quad \text{... (B)}$$

This is the required force exerted on $Q_2$ by $Q_1$.

# UNIT-II

# MAGNETOSTATICS

## Contents:

- Biot - Savart's Law
- Ampere's Circuital Law and Applications
- Magnetic Flux Density
- Maxwell's Equations for Magnetostatic Fields
- Magnetic Scalar and Vector Potentials
- Forces due to Magnetic Fields
- Ampere's Force Law
- Inductance and Magnetic Energy
- Illustrative Problem.

**Maxwell's Equations (Time Varying Fields):**

- Faraday's Law
- Transformer EMF
- Displacement Current Density
- Maxwell's Equations in Different Final Forms
- Conditions at a Boundary Surface: Dielectric - Dielectric,
- Illustrative Problems.

# Introduction:

In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated.

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later.

There are two major laws governing the magneto static fields are:

- Biot-Savart Law
- Ampere's Law

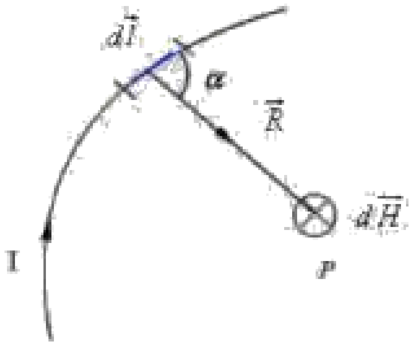
Usually, the magnetic field intensity is represented by the vector $\vec{H}$. It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 2.1.

$\vec{H}$ (or l ) out of the page $\vec{H}$ (or l ) into the page

Fig. Representation of magnetic field (or current)

## Biot- Savart's Law:

This law relates the magnetic field intensity dH produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig.



The magnetic field intensity $d\vec{H}$ at P can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$

$$dH = \frac{IdlSin\alpha}{4\pi R^2}$$

where $R = |\vec{R}|$ is the distance of the current element from the point P.
The value of the constant of proportionality 'K' depends upon a property called permeability of the medium around the conductor. Permeability is represented by symbol 'm' and the constant 'K' is expressed in terms of 'm' as

**Thus**

$$dB = \frac{\mu}{4\pi} \frac{I\,dl\;\sin\theta}{r^2}$$

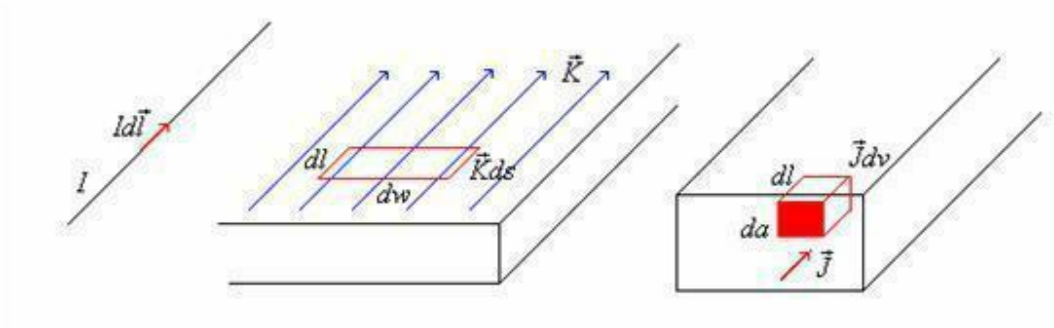
Magnetic field 'B' is a vector and unless we give the direction of 'dB', its description is not complete. Its direction is found to be perpendicular to the plane of 'r' and 'dl'.

If we assign the direction of the current 'I' to the length element 'dl', the vector product dl x r has magnitude r dl sinq and direction perpendicular to 'r' and 'dl'.

Hence, Biot–Savart law can be stated in vector form to give both the magnitude as well as direction of magnetic field due to a current element as

$$\vec{dB} = \frac{\mu}{4\pi} \frac{I\,(\vec{dl} \times \vec{r})}{r^3}$$

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 2.3.




Line Current Surface Current Volume Current

Fig. 2.3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m2) we can write:

$$Id\vec{l} = \vec{K}ds = \vec{J}dv$$

( It may be noted that $I = Kdw = Jda$ )

Employing Biot -Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions as

$$\vec{H} = \int_l \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$ ............................ for line current............................

$$\vec{H} = \int_S \frac{Kd\vec{s} \times \vec{R}}{4\pi R^3}$$ ........................ for surface current ....................

$$\vec{H} = \int_V \frac{\vec{J}dv \times \vec{R}}{4\pi R^3}$$ ....................... for volume current......................

## $\overline{H}$ Due to infinitely long straight conductor:

We consider a finite length of a conductor carrying a current $\vec{I}$ placed along z-axis as shown in the Fig 2.4. We determine the magnetic field at point P due to this current carrying conductor.

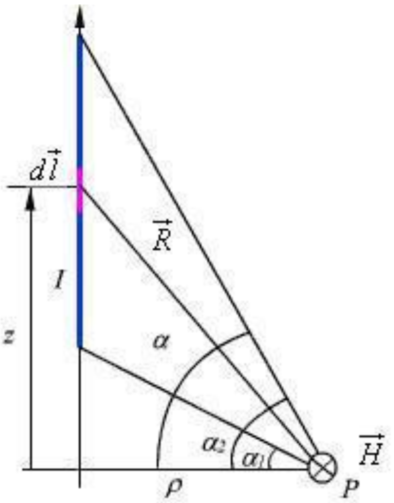



Fig. 2.4: Field at a point P due to a finite length current carrying conductor

With reference to Fig. 2.4, we find that

$$d\vec{l} = dz\,\hat{a}_z \quad and \quad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

Applying Biot - Savart's law for the current element $\vec{I}\,d\vec{l}$ We can write,

$$\overrightarrow{dH} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\rho dz \hat{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}}$$

Substituting $\frac{z}{\rho} = \tan\alpha$ we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi} \frac{\rho^2 \sec^2 \alpha d\alpha}{\rho^3 \sec^3 \alpha} \hat{a}_\phi = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\phi$$

We find that, for an infinitely long conductor carrying a current I , $\alpha_2 = 90^0$ and $\alpha_1 = -90^0$ Therefore

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

## **Ampere's Circuital Law**:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H ) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla\times\vec{H}.d\vec{s}$$

$$\therefore \int_S \nabla\times\vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s}$$

$$\therefore \nabla\times\vec{H} = \vec{J}$$

Which is the Ampere's circuital law in the point form and Maxwell’s equation for magneto static fields.

## Applications of Ampere's circuital law:

1. It is used to find $\bar{H}$ and $\bar{B}$ due to any type of current distribution.
2. If $\bar{H}$ or $\bar{B}$ is known then it is also used to find current enclosed by any closed path.

We illustrate the application of Ampere's Law with some examples.

## $\bar{H}$ Due to infinitely long straight conductor :( using Ampere's circuital law)

We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 2.5. Using Ampere's Law, we consider the close path to be a circle of radius $\rho$ as shown in the Fig. 4.5.

If we consider a small current element $Id\vec{l}(=Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(=\rho\hat{a}_\rho)$. Therefore only component of $\vec{H}$ that will be present is $H_\phi$,i.e., $\vec{H}=H_\phi\hat{a}_\phi$.

By applying Ampere's law we can write,

$$\vec{H}=\frac{I}{2\pi\rho}\hat{a}_\phi \quad \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi = I$$

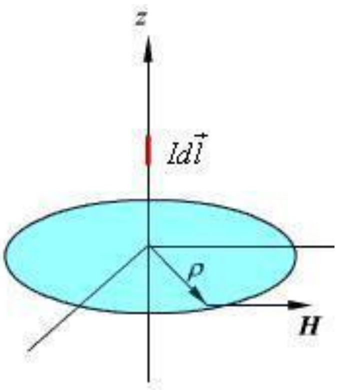



Fig. Magnetic field due to an infinite thin current carrying conductor

## $\bar{H}$ Due to infinitely long coaxial conductor :( using Ampere's circuital law)

We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 2.6. We compute the magnetic field as a function of $\rho$ as follows:

In the region $0 \le \rho \le R_1$

$$I_{enc} = I \frac{\rho^2}{R_1^2}$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2}$$

In the region $R_1 \le \rho \le R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

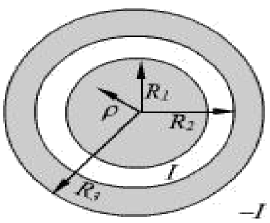



Fig. 2.6: Coaxial conductor carrying equal and opposite currents in the region

$R_2 \le \rho \le R_3$

$$H_\phi = \frac{I}{2\pi\rho}\frac{R_3^2 - \rho^2}{R_3^2 - R_2^2}$$

In the region $\rho > R_3$

$$I_{nec} = 0 \qquad H_\phi = 0$$

**Magnetic Flux Density:**

In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where $\mu$ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 = 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m 2 .

The magnetic flux density through a surface is given by:

$$\psi = \int_S \vec{B}.d\vec{s} \quad \text{Wb}$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 6 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

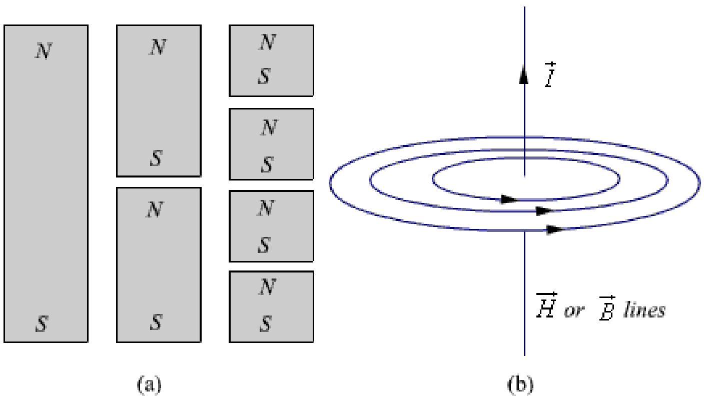



Fig. 6: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

## Maxwell's 2nd equation for static magnetic fields:

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 6 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$$\oint_s \vec{B}.d\vec{s} = 0$$ ......................................in integral form

which is the Gauss's law for the magnetic field.

By applying divergence theorem, we can write:

$$\oint_s \vec{B}.d\vec{s} = \int_v \nabla.\vec{B} dv = 0$$

Hence, $\nabla.\vec{B} = 0$ .................................. in point/differential form

which is the Gauss's law for the magnetic field in point form.

**Magnetic Scalar and Vector Potentials:**

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$$\vec{H} = -\nabla V_m$$

From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J}$$

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$ .

Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$ . Moreover, Vm in general is not a single valued function of position. This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 7.

In the region $a < \rho < b$ , $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

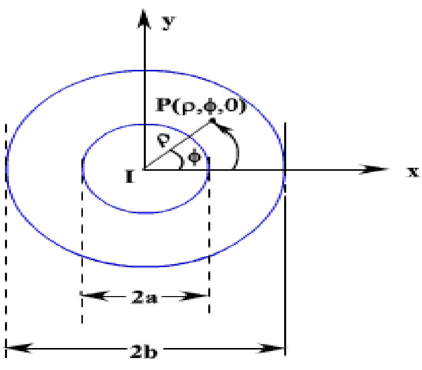


Fig. 7: Cross Section of a Coaxial Line

If Vm is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

If we set Vm = 0 at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach $\phi_0$ again but Vm this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of Vm keeps changing as we complete additional laps to pass through the same point. We introduced Vm analogous to electostatic potential V.

But for static electric fields,

$\nabla\times\vec{E} = 0$ and $\oint \vec{E}\cdot d\vec{l} = 0$

whereas for steady magnetic field $\nabla\times\vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H}\cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla.\vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla.\left(\nabla\times\vec{A}\right) = 0$ , we can write $\vec{B} = \nabla\times\vec{A}$ .

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation. We have introduced the vector function $\vec{B}$ and $\vec{A}$ related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla.\vec{A}$ is made as follows.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J}$$

By using vector identity, $\nabla \times \nabla \times \vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2 \vec{A}$

$$\nabla(\nabla.\vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}$$

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$ , we get $\nabla^2 \vec{A} = -\mu \vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x \quad .$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho}{R} dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$

$$\nabla.\vec{A} = \mu\varepsilon \frac{\partial V}{\partial t}$$

In case of time varying fields we shall see that , which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi} \int_{v'} \frac{J_x}{R} dv'$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J}}{R} dv'$$

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$.

Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'} \frac{I}{R} d\vec{l}\,'$$

$$\vec{A} = \frac{\mu}{4\pi}\int_{S'} \frac{\vec{K}}{R} ds\,'$$

.

The magnetic flux $\psi$ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s}$$

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A}.d\vec{s} = \oint_C \vec{A}.d\vec{l}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

# Forces due to magnetic fields

There are three ways in which the force due to magnetic fields can be experienced. The force can be

**(a) Force on a charged particle:**

We have $F_e$=QE

This shows that if Q is positive, $F_e$ and E are in same direction. It is found that the magnetic force $F_m$ experienced by a charge Q moving with a velocity u in magnetic field B is

$F_m$=Qu x B

For a moving change Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$F = F_e+F_m$

or

F=Q(E+u x B)

This is known as Lorentz force equation.

**(b) Force on a current element:**

To determine the force on a current element Idl of a current carrying conductor due to the magnetic field B, we take the equation

$J=P_e$ u

We have Idl= $\frac{dQ}{dt.}.dl = dQ = \frac{dl}{dt} = dQ.u$

Hence

Idl= dQ.u

This shows that an elemental charge dQ moving with velocity u (thereby producing convection current element dQu) is equivalent to a conduction current element Idl. Thus the force on current element is give by

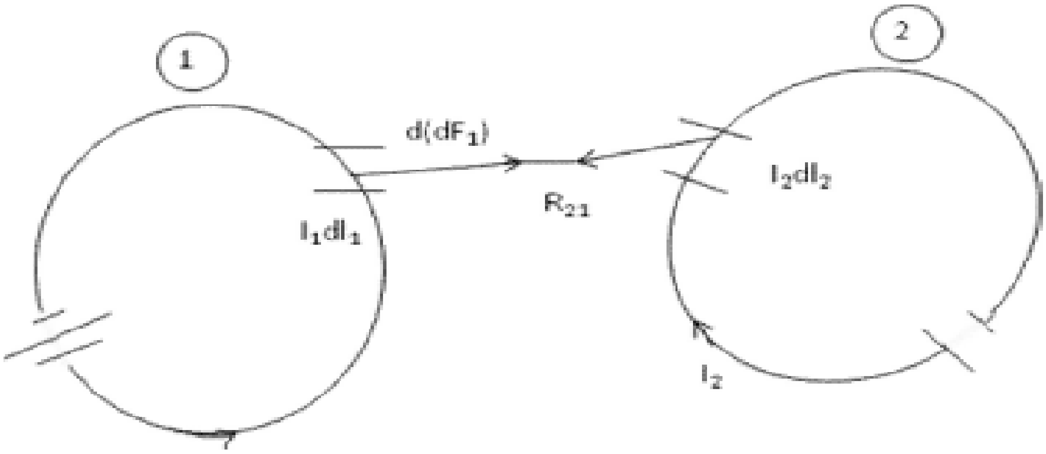
dF= Idl x B

If the current I is through a closed path L or circuit, the force on the circuit is given by

F= $\oint_L Idl \times B$

**(c) Force between two current elements:**

Consider the force between two elements $I_1dl_1$ and $I_2dl_2$. According to biotsavarts law, both current elements produce magnetic fields. Force $d(dF_1)$ on element $I_1dl_1$ due to field $dB_2$ produced by element $I_2\,dl_2$ as shown in figure below:




$d(dF_1) = I_1 Dl_1 \times dB_2$

But from biot Savarts law

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2}$$

*Hence*

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2}$$

This equation is the law of force between two current elements.

We have $F1 = \frac{\mu_0 I_1 I_2 \times a_{R21}}{4\pi} \oint_{L_1}\oint_{L_2} \frac{dI_1 \times (dI_2 \times aR_{21})}{R_{21}^2}$

## Inductance:

Inductance is the ability of the material to hold energy in form of magnetic field.

L, I are inductance of material and current flowing in the material.

$$E = \frac{1}{2} LI^2$$

$$\text{Inductance, L} = \frac{\text{Total flux linking current I}}{\text{current (I)}}$$

'B' is induced by I

$$\therefore \phi = \int_s \mathrm{B.d\bar{s}}$$

Total Flux depends on no of turns

Flux linking for n turns is 'Nϕ'.

$$\boxed{\therefore L = \frac{\lambda}{I}}$$

λ=Nϕ( depending on condition i.e total Flux linking the current)

## Inductance of a solenoid:

In the application of ampere's law to solenoid we found that

$$B = \frac{\mu NI}{l}\,Tesla$$

$$\therefore \phi = B.A = \frac{\mu NIA}{l}$$

With in a loop of N turns, the flux is linking the current N times.

$\therefore$ Total flux linking I = N$\phi$

$$= \frac{\mu N^2 IA}{l}$$

$$L = \frac{\lambda}{I} = \frac{\mu N^2 A}{l}$$

Some times inductors are given for unit length as well

$$\therefore \frac{l}{l} = \mu\left(\frac{N}{l}\right)^2 .A$$

**Inductance of coaxial cable:**

- The total flux linking the inner and outer conductors is same as the flux in the conductor.

$$H = \frac{I}{2\pi r}(A/m)$$

$$B = \frac{\mu I}{2\pi r}(Wb/m^2)$$

Here flux density is differing with radius

$$\therefore \phi = \oint \bar{B}.d\bar{s}$$

$$\therefore \phi = \int \frac{\mu I}{2\pi r} ds$$

$$d\bar{s} = dr dz \phi$$

$$\phi = \int_{z=0}^{L} \int_{r=a}^{b} \frac{\mu I}{2\pi r} dr dz$$

$$\phi = \frac{\mu I l}{2\pi} \int_a^b \frac{dr}{r}$$

$$\Rightarrow \lambda = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$\therefore L = \frac{\lambda}{I} = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$\frac{L}{l} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

Where μ is the permeability of medium used b/w inner and outer cores.

Also there is current flowing even inside the inner core.

$$= \frac{\mu I l \pi a^2}{8\pi} = \frac{\mu I l}{8\pi}$$

$$\therefore \frac{L_{\text{int}}}{l} = \frac{\mu}{8\pi}(H)$$

$$\frac{L_{ext}}{l} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)(H/m)$$

Here μ is permeability of conductor

$$\frac{\text{Total inductance}}{Length} = \frac{L_{ext}}{l} + \frac{L_{\text{int}}}{l}$$

$$= \frac{\mu_1}{2\pi}\ln\left(\frac{\text{b}}{\text{a}}\right) + \frac{\mu_2}{2\pi}$$

$$L = \frac{\Lambda}{I} = \frac{N\phi}{I}$$

To further illustrate the concept of inductance, let us consider two closed loops C1 and C2 as shown in the figure 8, S1 and S2 are respectively the areas of C1 and C2 .

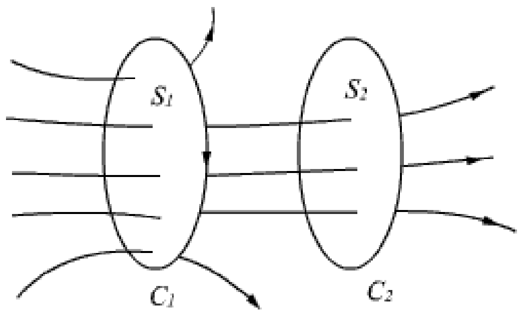



Fig:8

If a current I1 flows in C1 , the magnetic flux B1 will be created part of which will be linked to C2 as shown in Figure 8:

$$\phi_{12} = \int_{S_2} \vec{B}_1 . d\vec{S}_2$$

In a linear medium, $\phi_{12}$ is proportional to I 1. Therefore, we can write

$$\phi_{12} = L_{12} I_1$$

where L12 is the mutual inductance. For a more general case, if C2 has N2 turns then

$$\Lambda_{12} = N_2 \phi_{12}$$

and $\Lambda_{12} = L_{12} I_1$

or $$L_{12} = \frac{\Lambda_{12}}{I_1}$$

i.e., the mutual inductance can be defined as the ratio of the total flux linkage of the second circuit to the current flowing in the first circuit.

As we have already stated, the magnetic flux produced in C1 gets linked to itself and if C1 has N1 turns then $\Lambda_{11} = N_1 \phi_{11}$, where $\phi_{11}$ is the flux linkage per turn.

Therefore, self inductance

$$L_{11} \left( \text{or } L \text{ as defined earlier} \right) = \frac{\Lambda_{11}}{I_1}$$

As some of the flux produced by I1 links only to C1 & not C2.

$$\Lambda_{11} = N_1 \phi_{11} > N_2 \phi_{12} = \Lambda_{12}$$

Further in general, in a linear medium, $L_{12} = \frac{d\Lambda_{12}}{dI_1}$ and $L_{11} = \frac{d\Lambda_{11}}{dI_1}$

**Magnetic energy or Energy stored in Magnetic Field:**

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value I. As mentioned earlier, the self inductance of a coil in general can be written as

$$L = \frac{d\Lambda}{di} = N\frac{d\phi}{di}$$

or $$L\,di = N\,d\phi$$

If we consider a time varying scenario,

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

We will later see that $N\frac{d\phi}{dt}$ is an induced voltage.

$\therefore v = L\frac{di}{dt}$ is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$\begin{aligned} dW &= vi\,dt \\ &= Li\,di \end{aligned}$$

.

$$W = \int_0^I Li\,di = \frac{1}{2}LI^2 \qquad \text{(Joule)}$$

which is the energy stored in the magnetic circuit.

We can also express the energy stored in the coil in term of field quantities.

For linear magnetic circuit

$$W = \frac{1}{2}\frac{N\phi}{I}I^2 = \frac{1}{2}N\phi I$$

Now, $$\phi = \int_s \vec{B}.d\vec{S} = BA$$

where A is the area of cross section of the coil. If l is the length of the coil

$$NI = Hl$$

$$\therefore W = \frac{1}{2}HBAl$$

Al is the volume of the coil. Therefore the magnetic energy density i.e., magnetic energy/unit volume is given by

$$W_m = \frac{W}{Al} = \frac{1}{2}BH$$

In vector form

$$W_m = \frac{1}{2}\vec{B}.\vec{H} \qquad \text{J/mt3}$$

is the energy density in the magnetic field.

# UNIT-III

## MAXWELL'S EQUATIONS (Time varying Fields)

### Introduction:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \quad (1)$$

$$\nabla \cdot \vec{D} = \rho_v \quad (2)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \quad (3)$$

Similarly for the magnetostatic case

$$\nabla . \vec{B} = 0 \quad (4)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5)$$

$$\nabla \times \vec{H} = \vec{J} \quad (6)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and are therefore not generally introduced in an introductory treatment of the subject, except perhaps as summary relationships.

These basic equations of electricity and magnetism can be used as a starting point for advanced courses, but are usually first encountered as unifying equations after the study of electrical and magnetic phenomena.

| Symbols Used | | |
|---|---|---|
| E = Electric field | ρ = charge density | i = electric current |
| B = Magnetic field | ε0 = permittivity | J = current density |
| D = Electric displacement | μ0 = permeability | c = speed of light |
| H = Magnetic field strength | M = Magnetization | P = Polarization |

Integral form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$

Gauss' law for magnetism $\oint \vec{B} \cdot d\vec{A} = 0$

III. Faraday's law of induction $\oint \vec{E} \cdot \vec{ds} = -\frac{d\Phi_B}{dt}$

IV. Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E} \cdot d\vec{A}$$

Differential form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\nabla \cdot E = \dfrac{\rho}{\varepsilon_0} = 4\pi k\rho$

Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\dfrac{\partial B}{\partial t}$

IV. Ampere's law

$$\nabla\, x\; B = \frac{4\pi k}{c^2}J + \frac{1}{c^2}\frac{\partial E}{\partial t} = \frac{J}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$k = \frac{1}{4\pi\varepsilon_0} = \text{Coulomb's constant} \qquad c^2 = \frac{1}{\mu_0\varepsilon_0}$$

Differential form with magnetic and/or polarizable media:

I. Gauss' law for electricity $\nabla \cdot D = \rho$

$$D = \varepsilon_0 E + P \quad \text{General case}$$

$$D = \varepsilon_0 E \quad \text{Free space}$$

$$D = \varepsilon E \quad \text{Isotropic linear dielectric}$$

II. Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\dfrac{\partial B}{\partial t}$

IV. Ampere's law $\nabla\ x\ H = J + \dfrac{\partial D}{\partial t}$

$$B = \mu_0 (H + M) \quad \text{General case}$$

$$B = \mu_0 H \quad \text{Free space}$$

$$B = \mu H \quad \text{Isotropic linear magnetic medium}$$

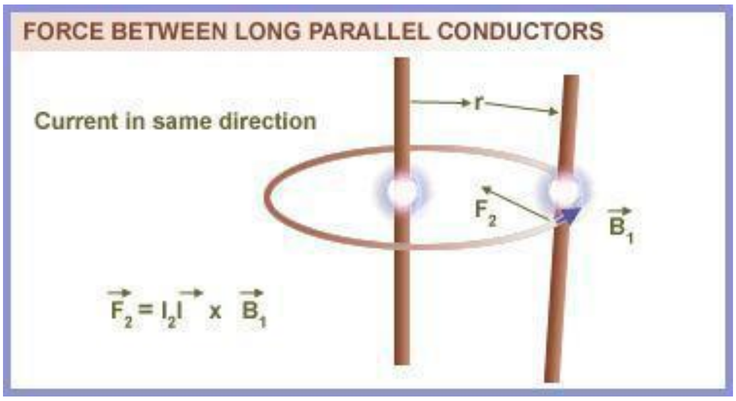
## Faraday's Law:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law.

Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

Faraday's law is a fundamental relationship which comes from Maxwell's equations. It serves as a succinct summary of the ways a voltage (or emf) may be generated by a changing magnetic environment. The induced emf in a coil is equal to the negative of the rate of change of magnetic flux times the number of turns in the coil. It involves the interaction of charge with magnetic field.

When two current carrying conductors are placed next to each other, we notice that each induces a force on the other. Each conductor produces a magnetic field around itself (Biot– Savart law) and the second experiences a force that is given by the Lorentz force.




Mathematically, the induced emf can be written as

$$\text{Emf} = -\frac{d\phi}{dt} \quad \text{Volts}$$

where $\phi$ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (7) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$\text{Emf} = -N\frac{d\phi}{dt} \quad \text{Volts}$$

By defining the total flux linkage as



$$\lambda = N\phi$$

The emf can be written as

Emf = $-\frac{d\lambda}{dt}$

Continuing with equation (3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.d\vec{l}$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_s \vec{B}.d\vec{s}$$

Where S is the surface for which 'C' is the contour.

From (11) and using (12) in (3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}.d\vec{s}$$

By applying stokes theorem

$$\int_s \nabla\times\vec{E}.d\vec{s} = -\int_s \frac{\partial\vec{B}}{\partial t}.d\vec{s}$$

Therefore, we can write

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

which is the Faraday's law in the point form

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf .

# Inconsistency of amperes law

Ampere's circuit law states that the line integral of tangential component of H around a closed path is same as the net current Ienc enclosed by the path.

i.e.

$$\int H.dl = I_{enc}$$

By applying stoke's theorem,

$$\int H.dl \, becomes \int_s J.ds$$

$\therefore$ Therefore, $\Delta \times H = J$ ______ (3.14)

This is true in case of static EM fields.

But in case of time-varying fields, the above Ampere's law shows same inconsistency.

**The inconsistency of ampere law for time varying fields is shown in two cases:**

1. For static EM fields, we have

$$\Delta \times H = J$$

Applying divergence on both sides, we get,

$$\Delta.(\Delta \times H) = \Delta.J$$

But divergence of curl of a vector field is always zero.

Therefore,

$$\Delta.(\Delta \times H) = 0 = \Delta.J$$

The continuity of current equation is given by

$$\Delta.J = \frac{-dp_v}{dt}$$

Where $J$ = Current density

$e_v$ = Charge density

For static fields, no current is produced, therefore, $e_v = 0 \Rightarrow \Delta.J = 0$

Implies eq. 3.15 is satisfied but for time varying fields, current is produced and therefore,

$$\Delta . J = \frac{-de_v}{dt} \# 0 \qquad (3.16)$$

Eq. (3.15) and eq. (3.16) are contradicting each other.

This is an inconsistency of ampere's law and the Ampere's law must be modified for time varying fields.

2. Consider the typical example of where the surface passes between the capacitor plates.

The figure is shown below.

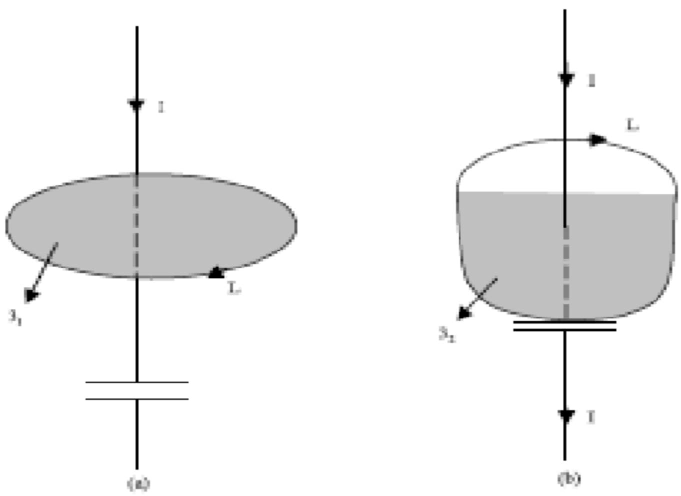



Fig 3.3 (a): Two surfaces of integration which explain the inconsistency of Ampere's law

In fig 3.3(a),

Based on Ampere's circuit law we get figure

$$\int_L H.dl = \int_{S_1} J.ds = I_{enc} = I \qquad (3.17)$$

In fig 3.3(b), based the ampere's circuit law, we get,

$$\oint_L H.dl = \int_{3_2} J.ds = I_{enc} = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.18)$$

Because no conduction current flows through $3_2$

i.e. J=0

in both (a) and (b), same closed path is used, but equations 3.17 and 3.18 are different.

This is an inconsistency of Ampere's circuit law.

This inconsistency of Ampere's circuit law in both cases (1) and (2) can be resolved by including displacement current in Ampere's circuit law.

Substituting in (3.19), we get,

$$\Delta \times H = J + \frac{dD}{dt} \_\_\_\_\_\_\_\_\_\_\_\_ (3.21)$$

This is the Maxwell equation (based on ampere's circuit Law) for tiem varying fields.

In equation (3.21),

$J_d$ = Displacement current density

$J$ = Conduction current density,

The conduction current density $J$ involves flow of charges. The displacement current density $J_d$ does not involve flow of charges. Displacement current,

$$I_d = \int Jd.ds = \int \frac{do}{dt}.ds \_\_\_\_\_\_\_\_\_\_\_ (3.22)$$

## Displacement Current Density:

The equation

$\Delta \times H = J$ For static EM fields is modified to Modified to

$$\Delta \times H = J + J_d \quad \underline{\hspace{4cm}} \quad (3.19)$$

To make the Ampere's law compatible for varying fields.

Now, applying divergence, we get

$$\Delta.(\Delta \times H) = 0 = \Delta.J + \Delta J_d$$

$$\Delta.J_d = -\Delta.J = \frac{de_v}{dt}$$

From Gauss Law, we have

$$e_v = \Delta.D$$

Therefore,

$$\Delta.J_d = \frac{d(\Delta.D)}{dt} = \Delta.\frac{dD}{dt}$$

$$\Rightarrow J_d = \frac{dD}{dt} \quad \underline{\hspace{3cm}} \quad (3.20)$$

## Boundary Condition for Magnetic Fields:

Similar to the boundary conditions in the electro static fields, here we will consider the behavior of $\vec{B}$ and $\vec{H}$ at the interface of two different media. In particular, we determine how the tangential and normal components of magnetic fields behave at the boundary of two regions having different permeabilities.

The figure 4.9 shows the interface between two media having permeabities $\mu_1$ and $\mu_2$, $\hat{a}_n$ being the normal vector from medium 2 to medium 1.

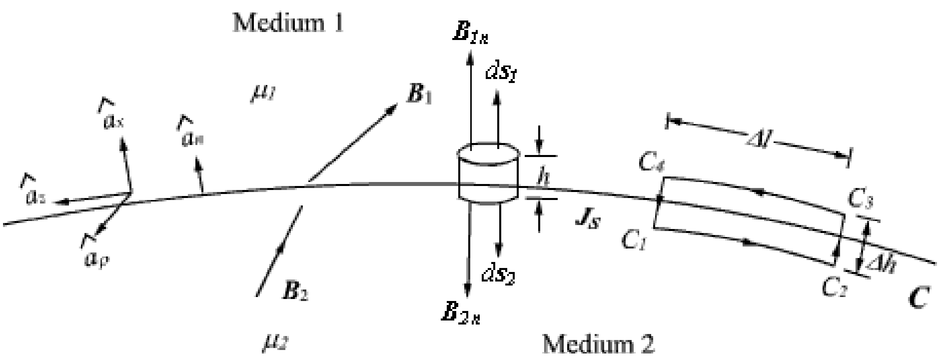



**Figure 4.9: Interface between two magnetic media**

o determine the condition for the normal component of the flux density vector $\vec{B}$, we consider a small pill box P with vanishingly small thickness *h* and having an elementary area $\Delta S$ for the faces. Over the pill box, we can write

$$\oint_S \vec{B}.d\vec{s} = 0 \qquad \text{.....................................................(4.36)}$$

Since h --> 0, we can neglect the flux through the sidewall of the pill box.

$$\therefore \int_{\Delta S} \vec{B}_1.d\vec{S}_1 + \int_{\Delta S} \vec{B}_2.d\vec{S}_2 = 0 \qquad \text{...........................(4.37)}$$

$$d\vec{S}_1 = dS\hat{a}_n \text{ and } d\vec{S}_2 = dS\left(-\hat{a}_n\right) \qquad \text{..................(4.38)}$$

$$\therefore \int_{\Delta S} B_{1n}dS - \int_{\Delta S} B_{2n}dS = 0$$

where

Since $\Delta S$ is small, we can write

$$\left(B_{1n} - B_{2n}\right)\Delta S = 0$$

or, $B_{1n} = B_{2n}$ ....................................(4.40)

That is, the normal component of the magnetic flux density vector is continuous across the interface.

In vector form,

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad \text{...........................(4.41)}$$

To determine the condition for the tangential component for the magnetic field, we consider a closed path C as shown in figure 4.8. By applying Ampere's law we can write

Since h -->0 , $\int_{c_1-c_2} \vec{H}.d\vec{l} + \int_{c_3-c_4} \vec{H}.d\vec{l} = I$

We have shown in figure 4.8, a set of three unit vectors $\hat{a}_n$, $\hat{a}_t$ and $\hat{a}_\rho$ such that they satisfy $\hat{a}_t = \hat{a}_\rho \times \hat{a}_n$ (R.H. rule). Here $\hat{a}_t$ is tangential to the interface and $\hat{a}_\rho$ is the vector perpendicular to the surface enclosed by C at the interface.

$$\oint \vec{H}.d\vec{l} = I$$

if $J_s = 0$, the tangential magnetic field is also continuous. If one of the medium is a perfect conductor $J_s$ exists on the surface of the perfect conductor.

In vector form we can write,

$$\left(\vec{H}_1 - \vec{H}_2\right).\hat{a}_t\,\Delta l$$

$$= \left(\vec{H}_1 - \vec{H}_2\right).\left(\hat{a}_\rho \times \hat{a}_n\right)\Delta l$$

$$= J_{Sn\Delta l} = \vec{J}_S.\hat{a}_\rho\,\Delta l$$

Therefore,

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S$$

# Solved problems:

## Problem1:

(a) In a cylindrical conductor to the region $0.01 \le r \le 0.02$, $0 < z < 1$ m and the current density is given by,

$$\vec{J} = 10e^{-100r}\hat{a}_\phi \text{ A/m}^2$$

Find the total current crossing the extential of this region with $\varphi$ = constant plane.

(b) Find the total current in a circular conductor of 4 mm radius if the current density varies according to $J = \frac{10^4}{r}$ A/m$^2$.

**Solution**

(a) Total current in the wire is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{r=0.01}^{0.02}\int_{z=0}^{1}\left[10e^{-100r}\hat{a}_\phi\right].\left[rdrdz\hat{a}_\phi\right]$$

$$= \int_{r=0.01}^{0.02}\int_{z=0}^{1} 10re^{-100r}drdz$$

$$= 10\int_{r=0.01}^{0.02} re^{-100r}dr$$

$$I = 10\left[\frac{re^{-100r}}{-100}\Bigg|_{0.01}^{0.02} - \int_{r=0.01}^{0.02}\frac{e^{-100r}}{-100}dr\right]$$

$$= 10\left[-\frac{1}{100}(0.02e^{-2} - 0.01e^{-1}) + \frac{e^{-100r}}{-100\times100}\Bigg|_{0.01}^{0.02}\right]$$

$$= 2\times10^{-3}e^{-1}$$

$$= 310^{-3}e^{-2}$$

(b) Total current is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{\phi=0}^{2\pi}\int_{r=0}^{0.004}\frac{10^4}{r}rdrd\phi = 2\pi\times10^4\int_{r=0}^{0.004}dr = 2\pi\times10^4\times0.004 = 80\pi \text{ A}$$

**Problem2:**

If $\vec{J} = \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)$ A/m$^2$, calculate the current passing through

(a) A hemispherical shell of 20 cm radius

(b) A spherical shell of 10 cm radius

**Solution**

Total current is given as $I = \int \vec{J}.d\vec{S}$

Here, $d\vec{S} = r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r$

(a) Total current passing through a hemispherical shell of 20 cm radius is,

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta).(r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r)\bigg|_{r=0.2}$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}2\cos\theta r^2\sin\theta\,d\phi\,d\theta\bigg|_{r=0.2}$$

$$= 2\pi \times \frac{2}{r}\int_{\theta=0}^{\pi/2}\sin\theta d(\sin\theta)\bigg|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 10\pi = 31.42\text{ A}$$

(b) Total current passing through a spherical shell of 10 cm radius is,

$$I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta).(r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r)\bigg|_{r=0.1}$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}2\cos\theta r^2\sin\theta\,d\phi\,d\theta\bigg|_{r=0.1}$$

$$= 2\pi \times \frac{2}{r}\int_{\theta=0}^{\pi}\sin\theta d(\sin\theta)\bigg|_{r=0.1}$$

$$= \frac{4\pi}{0.1}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi}$$

$$= 0$$

**Problem3:**

For the current density, $\vec{J} = 10z\sin^2\phi\, \hat{a}_r$ A/m$^2$, find the current through the cylindrical surface of $r = 2$, $1 \le z \le 5$ m.

**Solution**

Total current passing through the cylindrical surface is,

$$I = \int \vec{J}.d\vec{S} = \int_{z=1}^{5}\int_{\phi=0}^{2\pi} (10z\sin^2\phi\, \hat{a}_r).(rd\phi dz\hat{a}_r)\Big|_{r=2} = 10r\left[\frac{z^2}{2}\right]_1^5 \int_{\phi=0}^{2\pi} \sin^2\phi d\phi\Big|_{r=2}$$

$$= 10\times 2\times\frac{24}{2}\times\frac{2\pi}{2} = 240\pi = 754 \text{ A}$$

**Problem4:**

Determine the current density function $\vec{J}$ associated with the magnetic field defined by

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$ A/m (Cartesian)

(b) $\vec{H} = 6r\hat{a}_r + 2r\hat{a}_\phi + 5\hat{a}_z$ A/m (Cylindrical)

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\, \hat{a}_\phi$ A/m (Spherical)

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$

By Ampere's law in Cartesian coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 7 & 2x \end{vmatrix} = -2\hat{a}_y \text{ A/m}^2$$

(b) By Ampere's law in cylindrical coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \frac{1}{r}\hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \frac{1}{r}\left[\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial\phi}\right]\hat{a}_z$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial\phi}(5) - \frac{\partial}{\partial z}(2r)\right]\hat{a}_r + \left[\frac{\partial}{\partial z}(6r) - \frac{\partial}{\partial r}(5)\right]\hat{a}_\phi + \left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}(r2r) - \frac{\partial}{\partial\phi}(6r)\right]\hat{a}_z$$

$$= \left(\frac{1}{r}\right)\times 4r\hat{a}_z$$

$$= 4\hat{a}_z \text{ A/m}^2$$

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\, \hat{a}_\phi$

By Ampere's law in spherical coordinates,

$$\vec{J} = \nabla \times \vec{H} = \frac{1}{\rho^2 \sin\theta}\begin{vmatrix} \hat{a}_\rho & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_\rho & \rho H_\theta & \rho \sin\theta H_\phi \end{vmatrix}$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi}\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial H_\rho}{\partial \phi} - \frac{\partial}{\partial \rho}(\rho H_\phi)\right]\hat{a}_\theta$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\theta) - \frac{\partial H_\rho}{\partial \theta}\right]\hat{a}_\phi$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(\cos\theta \sin\theta) - \frac{\partial}{\partial \phi}(3)\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}(2\rho) - \frac{\partial}{\partial \rho}(\rho \cos\theta)\right]\hat{a}_\theta$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho 3) - \frac{\partial}{\partial \theta}(2\rho)\right]\hat{a}_\phi$$

$$= \frac{1}{\rho}\left(\frac{\cos 2\theta}{\sin\theta}\right)\hat{a}_\rho - \frac{1}{\rho}\cos\theta\, \hat{a}_\theta + \frac{3}{\rho}\hat{a}_\phi \text{ A/m}^2$$

**Problem5:**

An infinitely long conductor of radius a is placed such that its axis is along the *z*-axis. The vector magnetic potential, due to a direct current $I_0$ flowing along $\hat{a}_z$ in the conductor is given by

$$\vec{A} = -\frac{I_0}{4\pi a^2}\mu_0 (x^2 + y^2)\hat{a}_z \text{ Wb/m}$$

Find the corresponding $\vec{H}$. Also confirm the result using Ampere's law.

**Solution**

The magnetic flux density is given as,

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{I_0}{4\pi a^2}\mu_0 (x^2 + y^2) \end{vmatrix} = -\frac{I_0}{2\pi a^2}\mu_0 (y\hat{a}_x - x\hat{a}_y)$$

So, the magnetic field intensity is given as,

$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\frac{I_0}{2\pi a^2}(y\hat{a}_x - x\hat{a}_y)$$

We calculate the closed line integral of this field as follows.

$$\oint_L \vec{H}.d\vec{l} = -\frac{I_0}{2\pi a^2}\oint_L (y\hat{a}_x - x\hat{a}_y).(ad\phi\hat{a}_\phi) = -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(\hat{a}_\phi)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(-\sin\phi\hat{a}_x + \cos\phi\hat{a}_y)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(-y\sin\phi - x\cos\phi)$$

$$= \frac{I_0}{2\pi a^2}\oint_L ad\phi(a\sin^2\phi + a\cos^2\phi) \qquad \{\because x = r\cos\phi \quad \text{and} \quad y = r\sin\phi\}$$

$$= \frac{I_0}{2\pi}\oint_L d\phi(\sin^2\phi + \cos^2\phi)$$

$$= \frac{I_0}{2\pi}\oint_L d\phi = \frac{I_0}{2\pi}\times 2\pi = I_0$$

Since $\oint_L \vec{H}.d\vec{l} = I_0$, Ampere's law is verified.

**Problem6:**

Obtain an expression for the self-inductance of a toroid of circular section with '*N*' closely spaced turns.

**Solution**

Let,

$r$ = Mean radius of the toroid

$N$ = Number of turns

$S$ = Radius of the coil

We have the magnetic field,

$$H = \frac{NI}{2\pi r}$$

total flux linkage per turn is, $\phi = BA = \mu HA = \mu\frac{NI}{2\pi r}\pi S^2 = \frac{\mu NI}{2r}S^2$

Hence, the self-inductance of the toroid is $L = \frac{N\phi}{I} = \frac{\mu N^2 S^2}{2r}$

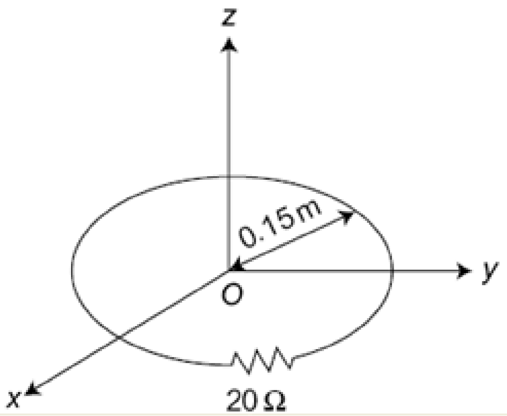
$$\boxed{L = \frac{\mu N^2 S^2}{2r}}$$

**Problem7:**

The circular loop conductor having a radius of 0.15 m is placed in the *xy* plane. This loop consists of a resistance of 20 Ω as shown in Fig. If the magnetic flux density is

$\vec{B} = 0.5 \sin 10^3 t\, \hat{a}_z$ **T**

Find the current flowing through the loop.




**Circular loop conductor**

**Solution**

Here since the loop is stationary and the magnetic field is time varying, only the transformer emf is induced.

Transformer emf induced is,

$$V_s = -\iint_S \frac{\partial \vec{B}}{\partial t} . d\vec{S} = -\iint_S \frac{\partial}{\partial t}(0.5 \sin 10^3 t \hat{a}_z).(r dr\, d\phi\, \hat{a}_z)$$

$$= -0.5 \times 10^3 \cos 10^3 t \int_{r=0}^{0.15} \int_{\phi=0}^{2\pi} r dr\, d\phi$$

$$= -0.5 \times 2\pi \times 10^3 \cos 10^3 t \left[ \frac{r^2}{2} \right]_0^{0.15}$$

$$= -10^3 \pi \cos 10^3 t \times 0.01125$$

$$= -35.34 \cos 10^3 t \text{ V}$$

**Problem8:**

(a) In free space, $\vec{D} = D_m \sin(\omega t + \beta z)\hat{a}_x$. Using Maxwell's equations, show that

$$\vec{B} = -\frac{\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

(b) In free space, $\vec{B} = B_m e^{j(\omega t + \beta z)}\hat{a}_y$. Using Maxwell's equations, show that

$$\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t + \beta z)}\hat{a}_x$$

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**Solution**

(a) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ and } \vec{D} = \varepsilon_0 \vec{E} \quad \text{or,} \quad \vec{E} = \frac{\vec{D}}{\varepsilon_0} \text{ for free space}$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{D_m}{\varepsilon_0}\sin(\omega t + \beta z) & 0 & 0 \end{vmatrix} = \frac{D_m}{\varepsilon_0}\frac{\partial}{\partial z}[\sin(\omega t + \beta z)]\hat{a}_y = \frac{D_m \beta}{\varepsilon_0}\cos(\omega t + \beta z)\hat{a}_y$$

$$\vec{B} = -\frac{D_m \beta}{\varepsilon_0}\int \cos(\omega t + \beta z)\hat{a}_y dt = -\frac{D_m \beta}{\omega \varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y$$

or,

Also, for free space,

$$\frac{\omega}{\beta} = v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad \frac{1}{\varepsilon_0} = \mu_0\left(\frac{\omega}{\beta}\right)^2$$

$$\vec{B} = -\frac{D_m \beta}{\omega \varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y = -\frac{D_m \beta}{\omega} \times \mu_0\left(\frac{\omega}{\beta}\right)^2 \sin(\omega t + \beta z)\hat{a}_y = -\frac{\omega \mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

$$\boxed{\vec{B} = -\frac{\omega \mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y}$$

(b) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} B_m e^{j(\omega t+\beta z)} \hat{a}_y$$

or, $$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$$

Comparing both sides, we get,

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$$

$$\frac{\partial E_x}{\partial z} = -B_m j\omega e^{j(\omega t+\beta z)} \qquad (\because E_z \text{ is not a function of } x)$$

$$E_x = \int -B_m j\omega e^{j(\omega t+\beta z)} dz = -B_m j\omega \frac{1}{j\beta} e^{j(\omega t+\beta z)} = -\frac{B_m \omega}{\beta} e^{j(\omega t+\beta z)}$$

$$\boxed{\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t+\beta z)} \hat{a}_x}$$

# UNIT – IV

## EM Wave Characteristics - I:

- Wave Equations for Conducting and Perfect Dielectric Media
- Uniform Plane Waves - Definition, Relation between E & H
- Wave Propagation in Lossless and Conducting Media
- Wave Propagation in Good Conductors and Good Dielectrics
- Illustrative Problems.

## EM Wave Characteristics - II:

- Reflection and Refraction of Plane Waves - Normal for both perfect Conductor and Perfect dielectric
- Brewster Angle
- Critical Angle
- Total Internal Reflection
- Surface Impedance
- Poynting Vector
- Poynting Theorem
- Illustrative Problems.

**Wave equations:**

The Maxwell's equations in the differential form are

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = \vec{\rho}$$

$$\nabla\cdot\vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant $\varepsilon$, magnetic permeability $\mu$ and conductivity $\sigma$. The above set of equations can be written as

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla\cdot\vec{E} = 0 \qquad (5.29(c))$$

$$\nabla\cdot\vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla\times\nabla\times\vec{A} = \nabla\cdot\left(\nabla\cdot\vec{A}\right) - \nabla^2 A$$

We can write from 2

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}$$

$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Substituting $\nabla\times\vec{H}$ from 1

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free( $\nabla\cdot\vec{E} = 0$) medium (eq3)

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

In the same manner for equation eqn 1

$$\nabla\times\nabla\times\vec{H} = \nabla\cdot\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H}$$

$$= \sigma\left(\nabla\times\vec{E}\right) + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H} = 0$ from eqn 4, we can write

$$\nabla^2\vec{H} = \mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla^2 \vec{H} = \mu\sigma \left( \frac{\partial \vec{H}}{\partial t} \right) + \mu\varepsilon \left( \frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

are known as wave equations.

**Uniform plane waves:**

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wave front or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios

Let us consider a plane wave which has only $E_x$ component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z

i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$ . The Helmholtz's equation reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

The solution to this equation can be written as

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$E_0^+ \ \& E_0^-$ are the amplitude constants (can be determined from boundary conditions).

In the time domain, $\varepsilon_X(z,t) = \mathrm{Re}(E_x(z) e^{jwt})$

$$\varepsilon_X(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

assuming $E_0^+ \ \& E_0^-$ are real constants.

Here, $\varepsilon_X^+(z,t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^+(z,t)$ for several values of t is shown in the Figure below

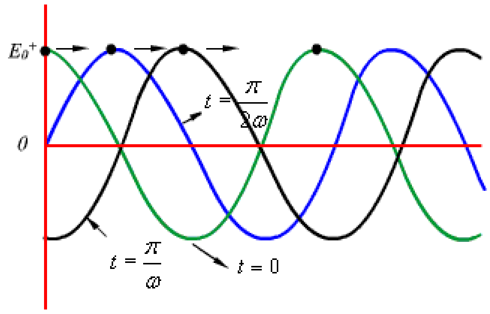



**Figure : Plane wave traveling in the + *z* direction**

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$ , z also should increase to $z + \Delta z$ so that

$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$

Or, $\omega \Delta t = k \Delta z$

Or, $\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$

When $\Delta t \to 0$ ,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity $v_p$ .

$\therefore v_p = \frac{\omega}{k}$

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$ , $\mu = \mu_0$

Then $v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength $\lambda$ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $(\omega t - kz) - [\omega t - k(z + \lambda)] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \frac{\omega}{v_p}$, $\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$

or, $\lambda f = v_p$

Thus wavelength $\lambda$ also represents the distance covered in one oscillation of the wave. Similarly, $\mathcal{E}^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.

When the wave travels in free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$$

is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^+(z,t) = \hat{a}_y\frac{E_0^+}{\eta}\cos(\omega t - \beta z)$$

Which represents the magnetic field of the wave traveling in the +z direction.

For the negative traveling wave,

$$\vec{H}^-(z,t) = -a_y\frac{E_0^+}{\eta}\cos(\omega t + \beta z)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig below

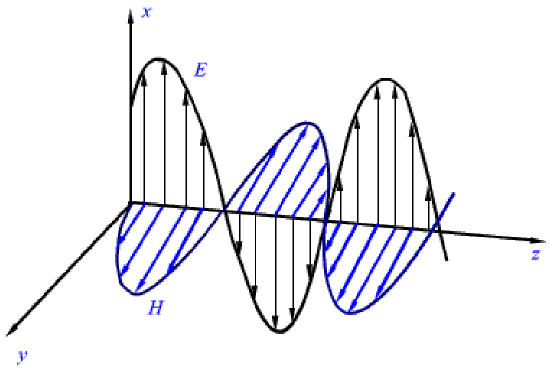



**Figure : E & H fields of a particular plane wave at time t.**

## Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$

the above curl equations we can write

$$\nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial\vec{D}}{\partial t} \quad \text{.............................................(1)}$$

In simple medium where $\in, \mu$ and $\sigma$ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right) \quad \text{and} \quad \vec{E}.\vec{J} = \sigma E^2$$

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV$$ ...........................(2)

The term $\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt$^2$) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

**Poynting vector for the time harmonic case:**

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\hat{a}_x$$ .................................(1)

when $E_0$ is real.
Let us consider two instanteneous quantities A and B such that

$$A = \mathrm{Re}\left(Ae^{j\omega t}\right) = |A|\cos(\omega t + \alpha)$$ ..................(2)

$$B = \mathrm{Re}\left(Be^{j\omega t}\right) = |B|\cos(\omega t + \beta)$$

where A and B are the phasor quantities.

i.e, $A = |A|e^{j\alpha}$

$B = |B|e^{j\beta}$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right]$$ ..............................(3)

Since $A$ and $B$ are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta) \qquad \text{.....................................(4)}$$

Further, considering the phasor quantities $A$ and $B$, we find that

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}(AB^*) \qquad \text{..............................................(5)}$$

The poynting vector $\vec{P} = \vec{E}\times\vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x(E_yH_z - E_zH_y) + \hat{a}_y(E_zH_x - E_xH_z) + \hat{a}_z(E_xH_y - E_yH_x) \qquad \text{...................................(6)}$$

If we consider a plane electromagnetic wave propagating in +z direction and has only $E_x$ component, from (6.42) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)\times H_y(z)\right) \qquad \text{........................................(7)}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.

For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) \qquad \text{.....................(8)}$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}(\vec{S})$.

## Polarization of plane wave:

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficient as the magnetic field components are related to electric field vector by the Maxwell's equations.

Let us consider a plane wave travelling in the +z direction. The wave has both $E_x$ and $E_y$ components.

$$\vec{E} = \left( \hat{a}_x E_{ox} + \hat{a}_y E_{oy} \right) e^{-j\beta z}$$

The corresponding magnetic fields are given by,

$$\begin{aligned}\vec{H} &= \frac{1}{\eta} \hat{a}_z \times \vec{E} \\ &= \frac{1}{\eta} \hat{a}_z \times \left( \hat{a}_x E_{ox} + \hat{a}_y E_{oy} \right) e^{-j\beta z} \\ &= \frac{1}{\eta} \left( -E_{oy} \hat{a}_x + E_{ox} \hat{a}_x \right) e^{-j\beta z}\end{aligned}$$

Depending upon the values of $E_{ox}$ and $E_{oy}$ we can have several possibilities:

1. If $E_{oy} = 0$, then the wave is linearly polarised in the $x$-direction.

2. If $E_{oy} = 0$, then the wave is linearly polarised in the $y$-direction.

3. If $E_{ox}$ and $E_{oy}$ are both real (or complex with equal phase), once again we get a linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1} \frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig1 below

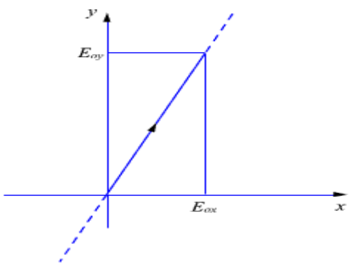



**Fig1 : Linear Polarisation**

4. If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}| e^{ja}$

$E_{oy} = |E_{oy}| e^{jb}$

Then,

$$E_x(z,t) = \mathrm{Re}\left[ |E_{ox}| e^{ja} e^{-j\beta z} e^{j\omega t} \right] = |E_{ox}| \cos(\omega t - \beta z + a)$$

and $E_y(z,t) = \text{Re}\left[\left|E_{oy}\right| e^{jb} e^{-j\beta z} e^{j\omega t}\right] = \left|E_{oy}\right| \cos(\omega t - \beta z + b)$ ....................................(2)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (2) we find that,

$$E_x(o,t) = \left|E_{ox}\right| \cos \omega t$$

$$E_y(o,t) = \left|E_{oy}\right| \cos\left(\omega t + \frac{\pi}{2}\right) = \left|E_{oy}\right|(-\sin \omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{\left|E_{ox}\right|}\right)^2 + \left(\frac{E_y(o,t)}{\left|E_{oy}\right|}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1 \quad ......................................(3)$$

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = \left|E_{ox}\right| \cos(\omega t)\hat{a}_x - \left|E_{oy}\right| \sin(\omega t)\hat{a}_y \quad .............................................(4)$$

Assuming $\left|E_{ox}\right| > \left|E_{oy}\right|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 2

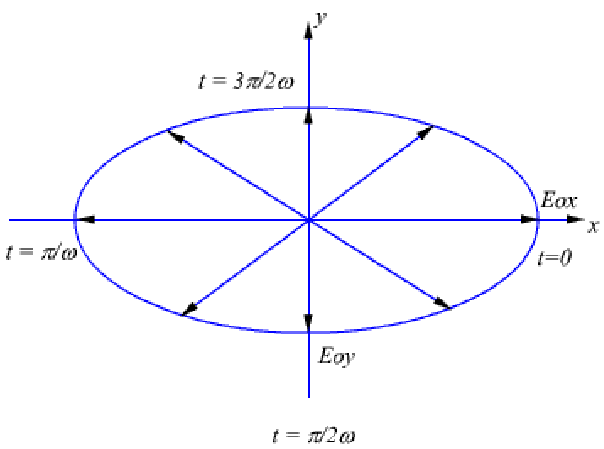



**Figure 2 : Plot of *E*(*o*,*t*)**

From equation (6.47) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces qn ellipse and the field is said to be elliptically polarised.

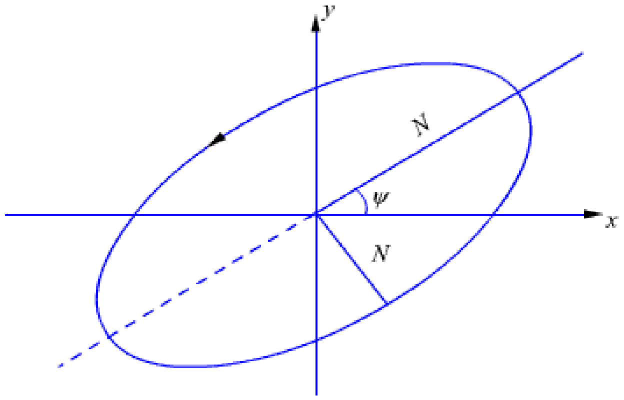



**Figure 3: Polarisation ellipse**

The polarisation ellipse shown in figure 3 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle $\psi$(orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW). Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity

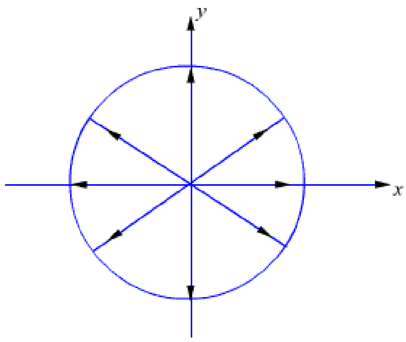



**Figure 5: Circular Polarisation (RHCP)**

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground( vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation ( one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

## Behaviour of Plane waves at the inteface of two media:

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of $\varepsilon, \mu, \sigma$ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

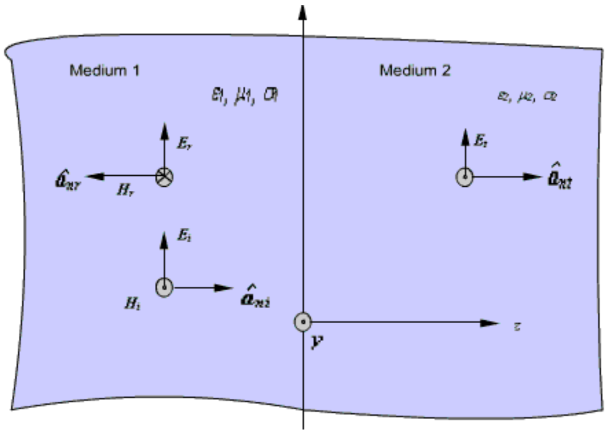



**Fig 6 : Normal Incidence at a plane boundary**

**Case1:** Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively. The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad ..................(1)$$

$$\vec{H}_i(z) = \frac{1}{\eta_i}\hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad ......................(2)$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$ .

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission.The reflected wave will travel along $\hat{a}_z$ in medium 1.

The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \quad .............................................(3)$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \quad .........(4)$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x$ ............................................(5)

$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y$ ............................................(6)

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2,

$\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$

& $\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$

From equation 3to 6 we get,

$E_{io} + E_{ro} = E_{to}$ .................................................................(7)

$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2}$ ..............................................................(8)

Eliminating $E_{to}$ ,

$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}(E_{io} + E_{ro})$

or, $E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$

or, $E_{ro} = \tau E_{io}$

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$ ...............(8)

is called the reflection coefficient.

From equation (8), we can write

$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$

or, $E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$

$T = \frac{2\eta_2}{\eta_1 + \eta_2}$ .......................................(9)

is called the transmission coefficient.

We observe that,

$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$ .......................................(10)

The following may be noted

(i) both $\tau$ and T are dimensionless and may be complex

(ii) $0 \le |\tau| \le 1$

Let us now consider specific cases:

**Case I: Normal incidence on a plane conducting boundary**

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$

$\eta_2 = 0$

$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$

$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$

From (9) and (10)

$\tau$ = -1

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$

& $\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$ ................................(11)

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t$ ..........................................................................(12)

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given *' t'*, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

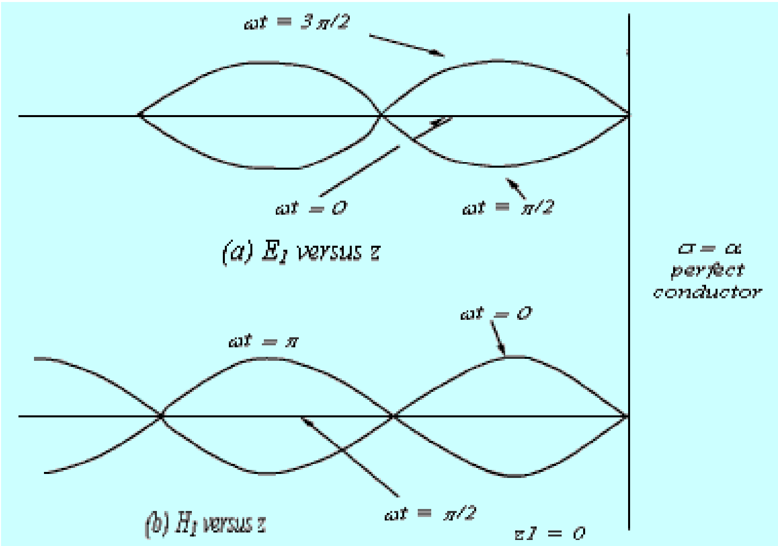



**Figure 7: Generation of standing wave**

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\right\} \text{occur at } \beta_1 z = -n\pi \qquad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \qquad \text{or } z = -(2n+1)\frac{\lambda}{4}, \quad n = 0,1,2,\ldots.$$

**Case2: Normal incidence on a plane dielectric boundary** : If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.

From equation (10) and equation (13) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad ..................(14)$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ( $\sigma_1 = 0,\ \sigma_2 = 0$ )

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad ..................(15)$$

In this case both $\eta_1$ and $\eta_2$ become real numbers.

$$\vec{E}_1 = \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)$$
$$= \hat{a}_x E_{io}\left(\left(1+T\right)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)$$
$$= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma\left(2j\sin\beta_1 z\right)\right) \qquad ..................(16)$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude $TE_{io}$ and a standing wave component with amplitude $2JE_{io}$. The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows. The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1+\Gamma e^{j2\beta_1 z}\right) ..................(17)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}\left(1+T\right) ..................(18)$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $z_{max} = -\dfrac{n\pi}{\beta_1} = -\dfrac{n\pi}{2\pi/\lambda_1} = -\dfrac{n}{2}\lambda_1$ , n = 0, 1, 2, 3.......................(19)

The minimum value of $\left|\vec{E}_1\right|$ is

$$\left|\vec{E}_1\right|_{min} = E_{io}\left(1-\Gamma\right) .................(20)$$

And this occurs when

$$2\beta_1 z_{min} = -\left(2n+1\right)\pi$$

or $z_{min} = -\left(2n+1\right)\dfrac{\lambda_1}{4}$ , n = 0, 1, 2, 3.............................(21)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1-\Gamma\right)$ which occurs at the $z_{min}$ locations and the minimum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1+\Gamma\right)$ which occurs at $z_{max}$ locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\dfrac{\left|E\right|_{max}}{\left|E\right|_{min}}$ can be written as

$$S = \frac{\left|E\right|_{max}}{\left|E\right|_{min}} = \frac{1+\left|\Gamma\right|}{1-\left|\Gamma\right|} .................(22)$$

The quantity $S$ is called as the standing wave ratio. As $0 \leq |\Gamma| \leq 1$ the range of $S$ is given by $1 \leq S \leq \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1 - \Gamma e^{j2\beta_1 z}\right) \quad \text{.................(23)}$$

From (6.68) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.

In medium 2, the transmitted wave propagates in the + z direction.

**Oblique Incidence of EM wave at an interface:** So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor.

ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

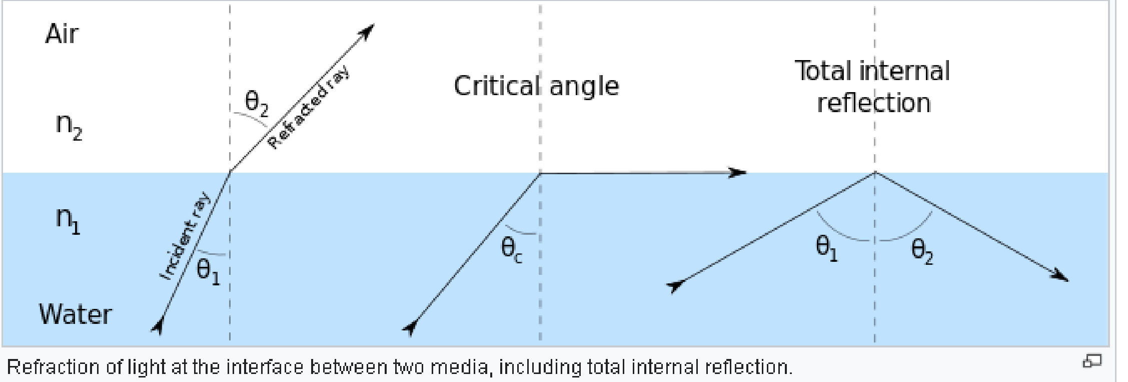
## Critical angle:

In geometric optics, at a refractive boundary, the smallest angle of incidence at which total internal reflection occurs. The critical angle is given by

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) ,$$

Where $\Theta_c$ is the critical angle, $n_1$ is the refractive index of the less dense medium, and $n_2$ is the refractive index of the denser medium.

**Angle of incidence:** The angle between an incident ray and the normal to a reflecting or refracting surface



Refraction of light at the interface between two media, including total internal reflection.

# UNIT-V

# TRANSMISSION LINES-I

- Types of transmission lines
- Transmission line Parameters- Primary & Secondary Constants
- Transmission Line Equations
- Expressions for Characteristics Impedance
- Propagation Constant
- Phase and Group Velocities
- Infinite Line Concepts
- Lossless transmission line
- Distortion
- Condition for Distortionlessness transmission
- Minimum Attenuation
- Illustrative Problems.

# TRANSMISSION LINE THEORY

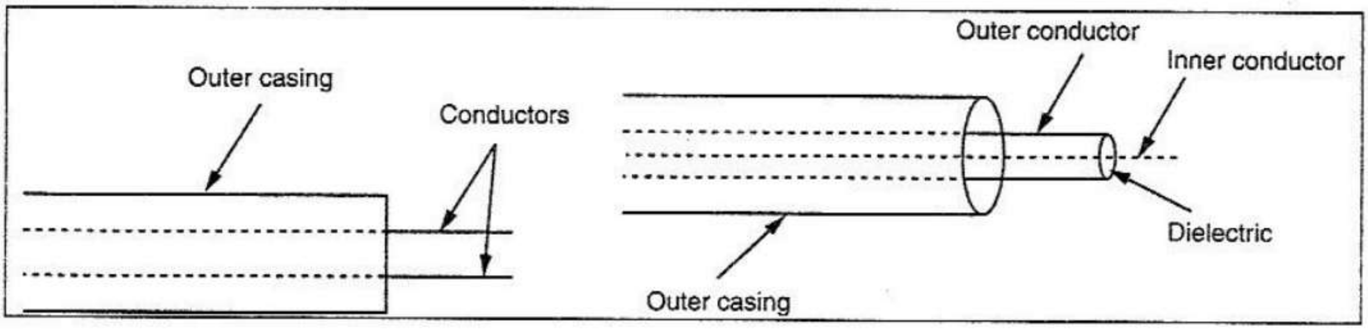
## 1.1. INTRODUCTION

The transfer of energy from one point to another takes place through either wave guides or transmission lines. Transmission lines always consist of atleast two separate conductors between which a voltage can exist, but the wave guides involve only one conductor; for example, a hollow rectangular or circular waveguide within which the wave propagates. Transmission lines are a means of conveying power from one point to another. There are two types of commonly used transmission lines.

1. Parallel wire (balanced) line
2. Coaxial (unbalanced) line

***Parallel wire line :*** It is a common form of transmission line known as open wire line as shown in Fig. 1.1(*a*). It is employed where balanced properties are required. Telephone lines, line connecting between folded dipole antenna and TV receiver are good examples of parallel or balanced or open wire line. The parallel wire lines are not used for microwave transmission.

***Coaxial line :*** Coaxial lines consist of inner and outer conductor spacers of dielectric as shown in Fig. 1.1(*b*). It is used when unbalanced properties are needed, as in the interconnection of a broadcast transmitter to its grounded antenna. It is employed at UHF and microwave frequencies.




*(a) Parallel wire (balanced) line* *(b) Coaxial (unbalanced) line*

*Fig. 1.1. Transmission lines*

## 1.2. TRANSMISSION LINE AS CASCADED T SECTIONS

To study the behaviour of transmission line, a transmission can be considered to be made up of a number of identical symmetrical T sections connected in series as in Fig.1.2. If the last section is terminated with its characteristic impedance, the input impedance at the first section is $Z_0$. Each section is terminated by the input impedance of the following section.

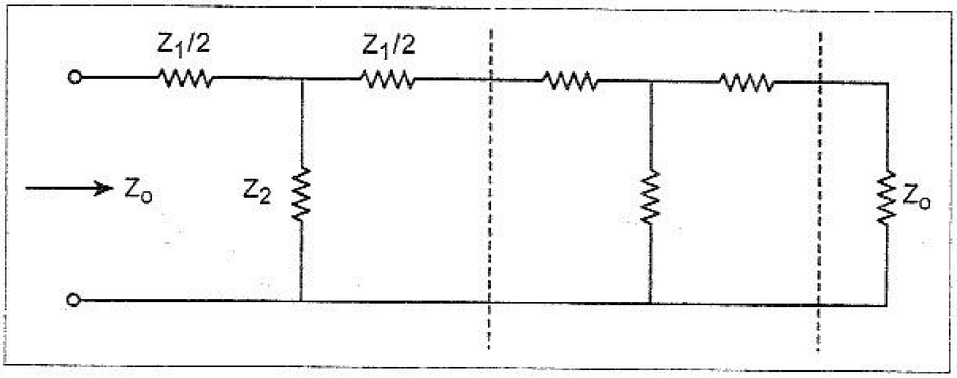



*Fig. 1.2. A line of cascaded T sections*

The characteristic impedance for a T section is

$$Z_{0T} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2}\right)}$$

If '$n$' number of T sections are cascaded and if the sending and receiving currents are $I_S$ and $I_R$ respectively, then

$$I_S = I_R \, e^{n\gamma}$$

where $\gamma$ is the propagation constant for one T section.

$$\gamma = \alpha + j\beta$$

$$e^{\gamma} = e^{\alpha + j\beta} = 1 + \frac{Z_1}{2 Z_2} + \sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4 Z_2}\right)}$$

One T section representing an incremental length $\Delta x$ of the line has a series impedance $Z_1 = Z \, \Delta x$ and shunt impedance $Z_2 = \frac{1}{Y \, \Delta x}$. The characteristic impedance of any small T section is that of the line as a whole.

$$Z_0 = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2}\right)}$$

Substituting the values of $Z_1$ and $Z_2$,

$$Z_0 = \sqrt{\frac{Z \, \Delta x}{Y \, \Delta x}\left(1 + \frac{Z \, \Delta x \, Y \, \Delta x}{4}\right)}$$

$$= \sqrt{\frac{Z}{Y}\left(1 + \frac{ZY \, (\Delta x)^2}{4}\right)}$$

If $\Delta x$ tends to zero, then $Z_0$ becomes,

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left(1+\frac{Z_1}{4Z_2}\right)^{\frac{1}{2}}$$

By the binomial theorem,

$$\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left[1+\frac{1}{2}\left(\frac{Z_1}{4Z_2}\right)-\frac{1}{8}\left(\frac{Z_1}{4Z_2}\right)^2+\ldots\ldots\right]$$

Substituting this value in $e^{\gamma}$ equation,

$$e^{\gamma} = 1+\frac{Z_1}{2Z_2}+\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)}$$

$$= 1+\frac{Z_1}{2Z_2}+\sqrt{\frac{Z_1}{Z_2}}+\frac{1}{8}\left(\frac{Z_1}{Z_2}\right)\sqrt{\frac{Z_1}{Z_2}}-\frac{1}{128}\left(\frac{Z_1}{Z_2}\right)^2\sqrt{\frac{Z_1}{Z_2}}+\ldots\ldots$$

$$= 1+\sqrt{\frac{Z_1}{Z_2}}+\frac{1}{2}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^2+\frac{1}{8}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^3-\frac{1}{128}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^5+\ldots\ldots$$

When applied to the incremental length of line $\Delta x$, then $Z_1 = Z\,\Delta x$, $Z_2 = \frac{1}{Y\,\Delta x}$ and propagation constant becomes $\gamma\,\Delta x$,

$$e^{\gamma\Delta x} = 1+\sqrt{ZY}\,\Delta x+\frac{1}{2}(\sqrt{ZY})^2(\Delta x)^2+\frac{1}{8}(\sqrt{ZY})^3(\Delta x)^3-128(\sqrt{ZY})^5(\Delta x)^5$$

Series expansion for an exponential $e^{\gamma\Delta x}$ is

$$e^{\gamma\Delta x} = 1+\gamma\Delta x+\frac{\gamma^2(\Delta x)^2}{2!}+\frac{\gamma^3(\Delta x)^3}{3!}+\ldots\ldots\ldots$$

Equating the above two expressions,

$$\sqrt{ZY}\,\Delta x+\frac{(\sqrt{ZY})^2(\Delta x)^2}{2}+\frac{(\sqrt{ZY})^2(\Delta x)^3}{8}+\ldots = \gamma\Delta x+\frac{\gamma^2(\Delta x)^2}{2}+\frac{\gamma^3(\Delta x)^3}{6}+\ldots$$

$$\gamma+\frac{\gamma^2\,\Delta x}{2}+\frac{\gamma^3(\Delta x)^2}{6}+\ldots\ldots = \sqrt{ZY}+\frac{(\sqrt{ZY})^2}{2}\Delta x+\frac{(\sqrt{ZY})^3(\Delta x)^2}{8}+\ldots\ldots$$

If $\Delta x$ tends to zero then,

$$\gamma = \sqrt{ZY}$$

This is the value of propagation constant in terms of Z and Y.

Since each conductor of transmission line has a certain length and diameter, it must have resistance and inductance; moreover the two conductors are separated by a dielectric medium (say, air), therefore there must be a capacitance between them. This dielectric between the conducting wires may not be perfect, and hence a leakage current will flow creating leakage (shunt) capacitance between the conductors. These four parameters resistance (R), inductance (L), capacitance (C) and conductance (G), all distributed along the lines are known as distributed parameters. The equivalent circuit diagram of transmission line is shown in Fig. 1.3.

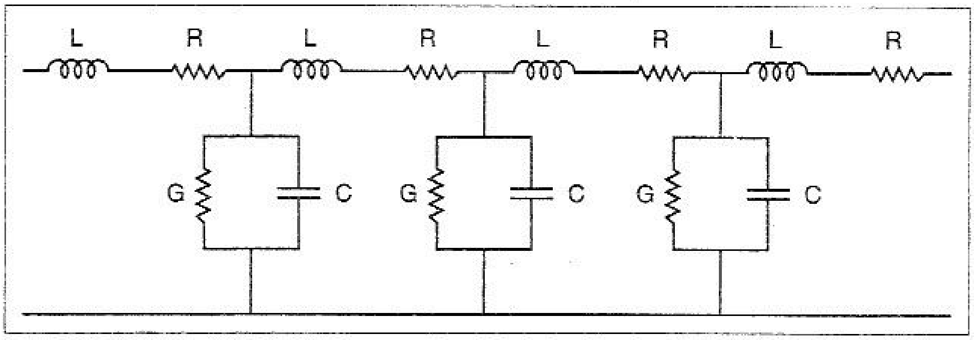



*Fig. 1.3. Equivalent circuit diagram of transmission line*

The four line parameters resistance (R), inductance (L), capacitance (C) and conductance (G) are also known as ***primary constants*** of the transmission line.

Resistance (R) is defined as the loop resistance per unit length of the transmission line. It is measured in ohms/km.

Inductance (L) is defined as the loop inductance per unit length of the transmission line. It is measured in Henries/km.

Capacitance (C) is defined as the shunt capacitance per unit length between the two transmission lines. It is measured in Farads/km.

Conductance (G) is defined as the shunt conductance per unit length between the two transmission lines. It is measured in mhos/km.

## 1.3. TRANSMISSION LINE EQUATION

Transmission line is a conductive method of guiding electrical energy from one place to another. A uniform transmission line can be considered to be made up of an infinite number of T sections, each of infinitesimal size $dx$. The equivalent circuit of T section of transmission line is shown in Fig. 1.4.

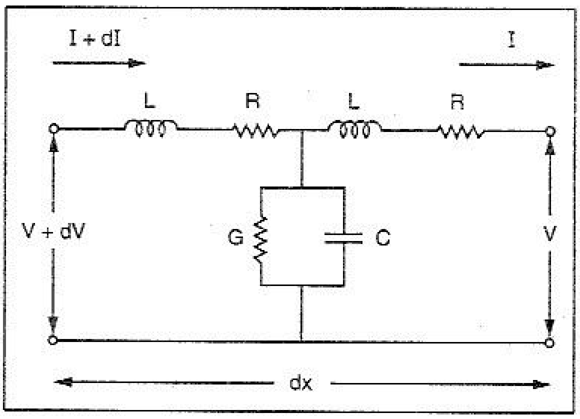



*Fig. 1.4. Equivalent circuit of T section of Transmission line*

The parameters R, L, G and C are distributed throughout the transmission line. The constants of an incremental length $dx$ of a line are shown in Fig. 1.4. The series impedance per unit length and shunt admittance per unit length are given by

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

Consider a T section of transmission line of length $dx$. Let V + $d$V be the voltage and I + $d$I be the current at one end of T section. Let V be the voltage and I be the current at the other end of this section.

The series impedance of a small section $dx$ is $(R + jL\omega)\, dx$. The shunt admittance of this section $dx$ is $(G + jC\omega)\, dx$.

The voltage drop across the series impedance of T sections *i.e.*, the potential difference between the two ends of T section is

$$V + dV - V = I\,(R + j\omega L)\, dx$$

$$dV = I\,(R + j\omega L)\, dx$$

$$\frac{dV}{dx} = I\,(R + j\omega L) \qquad \text{... (1.1)}$$

$$\frac{dV}{dx} = IZ$$

The current difference between the two ends of T section is due to the voltage drop across the shunt admittance.

$$I + dI - I = V\,(G + j\omega C)\, dx$$

$$dI = V\,(G + j\omega C)\, dx$$

$$\frac{dI}{dx} = V\,(G + j\omega C) \qquad \text{... (1.2)}$$

$$\frac{dI}{dx} = VY$$

Differentiating equation (1.1) w.r.t. '$x$',

$$\frac{d^2V}{dx^2} = (R + j\omega L)\,\frac{dI}{dx}$$

Substituting the value of $\frac{dI}{dx}$ in the above equation

$$\frac{d^2V}{dx^2} = (R + j\omega L)\,(G + j\omega C)\,V \qquad \text{... (1.3)}$$

Differentiating equation (1.2) w.r.t. '$x$'

$$\frac{d^2I}{dx^2} = (G + j\omega C)\,\frac{dV}{dx}$$

Substituting the value of $\frac{dV}{dx}$ in the above equation

$$\frac{d^2I}{dx^2} = (R + j\omega L)\,(G + j\omega C)\,I \qquad \text{... (1.4)}$$

But propagation constant is given by



$$\gamma = \sqrt{(R + j\omega L)\,(G + j\omega C)} = \sqrt{ZY}$$

Substituting the value of $\gamma$ in equation (1.3) and (1.4),

$$\text{then} \quad \frac{d^2V}{dx^2} = \gamma^2 V$$

$$\frac{d^2I}{dx^2} = \gamma^2 I$$

The solutions of the above linear differential equations are

$$V = A\, e^{\gamma x} + B\, e^{-\gamma x} \qquad \dots (1.5)$$

$$I = C\, e^{\gamma x} + D\, e^{-\gamma x} \qquad \dots (1.6)$$

where A, B, C and D are arbitrary constants.

Differentiating the equation (1.5), w.r.t. '$x$'

$$\frac{dV}{dx} = A\gamma\, e^{\gamma x} - B\gamma\, e^{-\gamma x}$$

$$\text{But} \quad \frac{dV}{dx} = IZ$$

$$IZ = A\gamma\, e^{\gamma x} - B\gamma\, e^{-\gamma x}$$

$$= A\sqrt{ZY}\, e^{\sqrt{ZY}\,x} - B\sqrt{ZY}\, e^{-\sqrt{ZY}\,x} \qquad [\because \gamma = \sqrt{ZY}\,]$$

$$I = A\sqrt{\frac{Y}{Z}}\, e^{\sqrt{ZY}\,x} - B\sqrt{\frac{Y}{Z}}\, e^{-\sqrt{ZY}\,x} \qquad \dots (1.7)$$

Similarly, differentiating the equation (1.6) w.r.t. '$x$'

$$\frac{dI}{dx} = C\gamma\, e^{\gamma x} - D\gamma\, e^{-\gamma x}$$

$$\text{But} \quad \frac{dI}{dx} = VY$$

$$VY = C\gamma\, e^{\gamma x} - D\gamma\, e^{-\gamma x}$$

$$= C\sqrt{ZY}\, e^{\sqrt{ZY}\,x} - D\sqrt{ZY}\, e^{-\sqrt{ZY}\,x}$$

$$V = C\sqrt{\frac{Z}{Y}}\, e^{\sqrt{ZY}\,x} - D\sqrt{\frac{Z}{Y}}\, e^{-\sqrt{ZY}\,x} \qquad \dots (1.8)$$

Since the distance $x$ is measured from the receiving end of the transmission line,

$$x = 0, \qquad \therefore \quad I = I_R$$

$$V = V_R$$



$$V_R = I_R Z_R$$

where $I_R$ is the current in the receiving end of line

$V_R$ is the voltage across the receiving end of the lines

$Z_R$ is the impedance of receiving end

Substituting this condition in equations (1.5), (1.6), (1.7) and (1.8).

$$V_R = A + B \quad \ldots (1.9)$$

$$I_R = C + D \quad \ldots (1.10)$$

$$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}} \quad \ldots (1.11)$$

$$V_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}} \quad \ldots (1.12)$$

To solve these equations,

$$\text{Let } x = \sqrt{\frac{Z}{Y}} \quad \text{and} \quad \frac{1}{x} = \sqrt{\frac{Y}{Z}}$$

$$\text{Then } I_R = \frac{A}{x} - \frac{B}{x}$$

$$= \frac{1}{x}(A - B)$$

$$\text{But } I_R = C + D$$

$$C + D = \frac{1}{x}(A - B)$$

$$Cx + Dx = A - B$$

$$A - B = Cx + Dx \quad \ldots (1.13)$$

Similarly, equation (1.12) becomes,

$$V_R = Cx - Dx$$

$$\text{But } V_R = A + B$$

$$A + B = Cx - Dx \quad \ldots (1.14)$$

$$A - B = Cx + Dx \quad \ldots (1.13)$$

Adding the equations (1.13) and (1.14),

$$2A = 2Cx$$

$$A = Cx$$

Similarly subtracting the equations (1.13) and (1.14),

$$2B = -2x\,Dx$$

$$B = -Dx$$

Substituting the values of A and B in the following equations.

$$V_R = A + B$$

$$= Cx - Dx$$

$$\text{But } I_R = C + D$$

$$I_R x = Cx + Dx \qquad \dots (1.15)$$

$$V_R = Cx - Dx \qquad \dots (1.16)$$

Adding the equations (1.15) and (1.16),

$$2Cx = I_R x + V_R$$

$$C = \frac{I_R}{2} + \frac{V_R}{2x}$$

$$\boxed{\therefore C = \frac{I_R}{2} + \frac{V_R}{2}\sqrt{\frac{Y}{Z}}} \qquad \dots (1.17) \qquad \left[\because x = \sqrt{\frac{Z}{Y}}\right]$$

Subtracting the equations (1.15) and (1.16),

$$2Dx = I_R x - V_R$$

$$D = \frac{I_R}{2} - \frac{V_R}{2x}$$

$$\boxed{\therefore D = \frac{I_R}{2} - \frac{V_R}{2}\sqrt{\frac{Y}{Z}}} \qquad \dots (1.18)$$

$$\text{But } A = Cx$$

$$A = \frac{I_R}{2}x + \frac{V_R}{2}$$

$$\boxed{\therefore A = \frac{V_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}}} \qquad \dots (1.19)$$

$$B = -Dx$$

$$B = -\frac{I_R}{2}x + \frac{V_R}{2}$$

$$\boxed{\therefore B = \frac{V_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}}} \qquad \dots (1.20)$$

The characteristic impedance is defined as

$$Z_o = \sqrt{\frac{Z}{Y}}$$

$$= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \dots (1.21)$$

Substituting the value of $Z_o$ in equations (1.19), (1.20), (1.17) and (1.18),

$$A = \frac{V_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}}$$

$$A = \frac{V_R}{2} + \frac{V_R}{2\,Z_R}\,Z_o$$

$$\boxed{A = \frac{V_R}{2}\left[1 + \frac{Z_o}{Z_R}\right]} \qquad \ldots (1.22)$$

$$B = \frac{V_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}}$$

$$= \frac{V_R}{2} - \frac{V_R}{2\,Z_R}\,Z_o$$

$$\boxed{B = \frac{V_R}{2}\left[1 - \frac{Z_o}{Z_R}\right]} \qquad \ldots (1.23)$$

$$C = \frac{I_R}{2} + \frac{V_R}{2}\sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} + \frac{I_R\,Z_R}{2\,Z_o} \qquad [\because V_R = I_R\,Z_R]$$

$$\boxed{C = \frac{I_R}{2}\left[1 + \frac{Z_R}{Z_o}\right]} \qquad \ldots (1.24)$$

$$D = \frac{I_R}{2} - \frac{V_R}{2}\sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} - \frac{I_R\,Z_R}{2\,Z_o}$$

$$\boxed{D = \frac{I_R}{2}\left[1 + \frac{Z_R}{Z_o}\right]} \qquad \ldots (1.25)$$

Substituting the values of A, B, C and D in equations (1.5) and (1.6), the solutions of the differential equations are

$$V = \frac{V_R}{2}\left(1 + \frac{Z_o}{Z_R}\right) e^{\sqrt{ZY}\,x} + \frac{V_R}{2}\left(1 - \frac{Z_o}{Z_R}\right) e^{-\sqrt{ZY}\,x} \qquad \ldots (1.26)$$

$$I = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_o}\right) e^{\sqrt{ZY}\,x} + \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_o}\right) e^{-\sqrt{ZY}\,x} \qquad \ldots (1.27)$$

$$V = \frac{V_R}{2}\left[\left(1 + \frac{Z_o}{Z_R}\right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_o}{Z_R}\right) e^{-\sqrt{ZY}\,x}\right] \qquad \ldots (1.28)$$

$$I = \frac{I_R}{2}\left[\left(1+\frac{Z_R}{Z_o}\right)e^{\sqrt{ZY}\,x}+\left(1-\frac{Z_R}{Z_o}\right)e^{-\sqrt{ZY}\,x}\right] \quad \ldots (1.29)$$

After simplification,

$$V = \frac{V_R}{2}e^{\sqrt{ZY}\,x}+\frac{V_R}{2}\frac{Z_o}{Z_R}e^{\sqrt{ZY}\,x}+\frac{V_R}{2}e^{-\sqrt{ZY}\,x}-\frac{V_R}{2}\frac{Z_o}{Z_R}e^{-\sqrt{ZY}\,x}$$

$$I = \frac{I_R}{2}e^{\sqrt{ZY}\,x}+\frac{I_R}{2}\frac{Z_R}{Z_o}e^{\sqrt{ZY}\,x}+\frac{I_R}{2}e^{-\sqrt{ZY}\,x}-\frac{I_R}{2}\frac{Z_R}{Z_o}e^{-\sqrt{ZY}\,x}$$

$$V = V_R\left(\frac{e^{\sqrt{ZY}\,x}+e^{-\sqrt{ZY}\,x}}{2}\right)+I_R Z_o\left(\frac{e^{\sqrt{ZY}\,x}-e^{-\sqrt{ZY}\,x}}{2}\right) \quad [\because V_R = I_R Z_R]$$

$$I = I_R\left(\frac{e^{\sqrt{ZY}\,x}+e^{-\sqrt{ZY}\,x}}{2}\right)+\frac{V_R}{Z_o}\left(e^{\sqrt{ZY}\,x}-e^{-\sqrt{ZY}\,x}\right) \quad \left[\because I_R = \frac{V_R}{Z_R}\right]$$

Then equations can be written in terms of hyperbolic functions.

$$\boxed{\begin{aligned} V &= V_R\cosh\sqrt{ZY}\;x+I_R Z_o\sinh\sqrt{ZY}\;x \quad \ldots (1.30)\\ I &= I_R\cosh\sqrt{ZY}\;x+\frac{V_R}{Z_o}\sinh\sqrt{ZY}\;x \quad \ldots (1.31)\end{aligned}}$$

These are the equations for voltage and current of a transmission line at any distance '$x$' from the receiving end of transmission line.

The equations for voltage and current at the sending send of a transmission line of length '$l$' are given by

$$V_S = V_R\cosh\sqrt{ZY}\,l+\frac{V_R}{Z_R}Z_o\sinh\sqrt{ZY}\,l \quad \left[\because I_R = \frac{V_R}{Z_R}\right]$$

$$I_S = I_R\cosh\sqrt{ZY}\,l+\frac{I_R Z_R}{Z_o}\sinh\sqrt{ZY}\,l \quad [\because V_R = I_R Z_R]$$

$$V_S = V_R\left[\cos\sqrt{ZY}\,l+\frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right] \quad \ldots (1.32)$$

$$I_S = I_R\left[\cos\sqrt{ZY}\,l+\frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right] \quad \ldots (1.33)$$

## 1.4. WAVELENGTH AND VELOCITY OF PROPAGATION

The propagation constant ($\gamma$) and characteristic impedance ($Z_0$) are called secondary constants of a transmission line.



Propagation constant is usually a complex quantity.



$$\gamma = \alpha + j\beta$$

where $\alpha$ is the attenuation constant.

$\beta$ is the phase shift.

$$\gamma = \sqrt{ZY}$$

where
$$Z = R + j\omega L$$
$$Y = G + j\omega C$$

The characteristic impedance of the transmission line is also a complex quantity.

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \dots (1.34)$$

Propagation constant is
$$\gamma = \alpha + i\beta$$
$$= \sqrt{(R + j\omega L)(G + j\omega C)}$$
$$\alpha + i\beta = \sqrt{RG - \omega^2 LC + j\omega(LG + RC)} \quad \dots (1.35)$$

Squaring on both sides,

$$(\alpha + j\beta)^2 = RG - \omega^2 LC + j\omega(LG + RC)$$
$$\alpha^2 - \beta^2 + 2j\,\alpha\beta = RG - \omega^2 LC + j\omega\,(LG + RC) \quad \dots (1.36)$$

Equating real parts,

$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$
$$\alpha^2 = \beta^2 + RG - \omega^2 LC \quad \dots (1.37)$$

Equating imaginary parts,

$$2\,\alpha\beta = \omega\,(LG + RC)$$

Squaring on both sides,

$$4\,\alpha^2\beta^2 = \omega^2\,(LG + RC)^2$$
$$\alpha^2\beta^2 = \frac{\omega^2}{4}\,(LG + RC)^2$$

Substituting the value of $\alpha^2$ [eqn. (1.37)] in the above equation,

$$(\beta^2 + RG - \omega^2 LC)\,\beta^2 = \frac{\omega^2}{4}\,(LG + RC)^2$$

$$\beta^4 + \beta^2\,(RG - \omega^2 LC) - \frac{\omega^2}{4}\,(LG + RC)^2 = 0$$

The solution of the quadratic equation is



$$\beta^2 = \frac{-(RG - \omega^2 LC) \pm \sqrt{(RG - \omega^2 LC)^2 + \omega^2\,(LG + RC)^2}}{2}$$

By neglecting the negative values,

$$\therefore \quad \beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \quad \ldots (1.38)$$

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \quad \ldots (1.37)$$

Substituting the value of $\beta$ [eqn. (1.38)] in the above equation,

$$\alpha^2 = \frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2} + RG - \omega^2 LC$$

$$= \frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}$$

$$\therefore \quad \alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \quad \ldots (1.39)$$

For a perfect transmission line $R = 0$ and $G = 0$,

$$\beta^2 = \omega^2 LC$$

$$\therefore \quad \beta = \omega \sqrt{LC} \qquad \text{[only positive value]}$$

***Velocity :***

The velocity of propagation is given by,

$$v = \lambda f$$

$$= 2\pi f \frac{\lambda}{2\pi}$$

$$v = \frac{\omega}{\beta} \qquad \left[\because \beta = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi f\right]$$

Substituting the value of $\beta = \omega \sqrt{LC}$

$$\therefore \quad v = \frac{\omega}{\omega \sqrt{LC}}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the velocity of propagation for an ideal line.

***Wavelength :***

The distance travelled by the wave along the line while the phase angle is changing through $2\pi$ radians is called wavelength.

$$\beta\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{\beta} \quad \text{or} \quad \lambda = \frac{v}{f}$$

## 1.5. INPUT IMPEDANCE AND TRANSFER IMPEDANCE OF TRANSMISSION LINE

***Input impedance :***

The equations for voltage and current at the sending end of a transmission line of length '$l$' are given by

$$V_S = V_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right) \quad \dots (1.32)$$

$$I_S = I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right) \quad \dots (1.33)$$

The input impedance of the transmission line is,

$$Z_S = \frac{V_S}{I_S}$$

$$= \frac{V_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right)}{I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right)}$$

$$= \frac{I_R Z_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right)}{I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right)}$$

$$Z_S = \frac{Z_0\,(Z_R\cosh\sqrt{ZY}\,l + Z_0\sinh\sqrt{ZY}\,l)}{(Z_0\cosh\sqrt{ZY}\,l + Z_R\sinh\sqrt{ZY}\,l)} \quad \dots (1.40)$$

Let $\sqrt{ZY} = \gamma$

The input impedance of the line is

$$Z_S = Z_0\left[\frac{Z_R\cosh\gamma l + Z_0\sinh\gamma l}{Z_0\cosh\gamma l + Z_R\sinh\gamma l}\right]$$

or $$Z_S = Z_0\left[\frac{Z_R + Z_0\tanh\gamma l}{Z_0 + Z_R\tanh\gamma l}\right]$$

In a different form, the equations for voltage and current at transmitting end of a line is given by equations (1.28) and (1.29),

$$V_S = \frac{V_R}{2}\left[\left(1 + \frac{Z_0}{Z_R}\right)e^{\sqrt{ZY}\,l} + \left(1 - \frac{Z_0}{Z_R}\right)e^{-\sqrt{ZY}\,l}\right] \quad \dots (1.28)$$

$$I_S = \frac{I_R}{2}\left[\left(1 + \frac{Z_R}{Z_0}\right)e^{\sqrt{ZY}\,l} + \left(1 - \frac{Z_R}{Z_0}\right)e^{-\sqrt{ZY}\,l}\right] \quad \dots (1.29)$$

or $$V_S = \frac{V_R}{2}\left[\left(\frac{Z_R+Z_0}{Z_R}\right) e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R}\right) e^{-\sqrt{ZY}\,l}\right]$$

$$I_S = \frac{I_R}{2}\left[\left(\frac{Z_R+Z_0}{Z_0}\right) e^{\sqrt{ZY}\,l} + \left(\frac{Z_0-Z_R}{Z_0}\right) e^{-\sqrt{ZY}\,l}\right]$$

or $$V_S = \left(\frac{V_R}{2}\right)\left(\frac{Z_R+Z_0}{Z_R}\right)\left[e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\sqrt{ZY}\,l}\right] \quad \text{... (1.41)}$$

$$I_S = \frac{I_R}{2}\left(\frac{Z_R+Z_0}{Z_0}\right)\left[e^{\sqrt{ZY}\,l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\sqrt{ZY}\,l}\right] \quad \text{... (1.42)}$$

The input impedance of the transmission line is given by,

$$Z_S = \frac{V_S}{I_S} = Z_0\left[\frac{e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\sqrt{ZY}\,l}}{e^{\sqrt{ZY}\,l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\sqrt{ZY}\,l}}\right] \quad [\because V_R = I_R Z_R] \quad \text{... (1.43)}$$

Let $\sqrt{ZY} = \gamma$

The input impedance of the transmission line is,

$$Z_S = Z_0\left[\frac{e^{\gamma l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\gamma l}}{e^{\gamma l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right) e^{-\gamma l}}\right] \quad \text{... (1.44)}$$

If the line is terminated with its characteristic impedance *i.e.*, $Z_R = Z_0$, then the input impedance becomes equal to its characteristic impedance.

$$Z_S = Z_0$$

The input impedance of an infinite line is determined by letting $l \to \infty$.

$$\therefore \; Z_S = Z_0$$

It is found that a line of finite length, terminated with its characteristic impedance, appears to the transmitting end generator as an infinite line. A finite line terminated with $Z_0$ and an infinite line are same by measurements at the source.

If $K = \dfrac{Z_R-Z_0}{Z_R+Z_0}$, then

$$Z_S = Z_0\left[\frac{e^{\gamma l} + K e^{-\gamma l}}{e^{\gamma l} - K e^{-\gamma l}}\right] \quad \text{... (1.45)}$$

***Transfer impedance :***

Transfer impedance is used to determine the current at the receiving end if voltage at transmitting end is known. Transfer impedance of a transmission line is defined as the ratio of voltage at the sending end (transmitted voltage) to the current at the receiving end (received current).

$$Z_T = \frac{V_S}{I_R}$$

Equation (1.41) becomes

$$V_S = \frac{V_R (Z_R + Z_0)}{2 Z_R} \left(e^{\gamma l} + K e^{-\gamma l}\right)$$

$$V_S = \frac{I_R (Z_R + Z_0)}{2} \left(e^{\gamma l} + K e^{-\gamma l}\right) \qquad [\because V_R = I_R Z_R]$$

$$Z_T = \frac{V_S}{I_R} = \frac{Z_R + Z_0}{2} \left(e^{\gamma l} + K e^{-\gamma l}\right)$$

$$= \frac{Z_R + Z_0}{2} \left(e^{\gamma l} + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\gamma l}\right)$$

$$= \left(\frac{Z_R + Z_0}{2}\right) e^{\gamma l} + \left(\frac{Z_R - Z_0}{2}\right) e^{-\gamma l}$$

$$= Z_R \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2}\right) + Z_0 \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2}\right)$$

$$= Z_R \cosh \gamma l + Z_0 \sinh \gamma l$$

$$\left[\because \frac{e^{\gamma l} + e^{-\gamma l}}{2} = \cosh \gamma l \quad \text{and} \quad \frac{e^{\gamma l} - e^{-\gamma l}}{2} = \sinh \gamma l\right]$$

$$Z_T = Z_R \cosh \gamma l + Z_0 \sinh \gamma l$$

## 1.6. LINE DISTORTION

Signal (*e.g.*, voice) transmitted over a transmission line is normally complex and consists of many frequency components. Such voice voltage will not have all frequencies transmitted with equal attenuation and equal time delay, the received waveform will not be identical with the input waveform at the sending end. This variation is known as ***distortion***. There are two types of line distortions. They are frequency distortion and delay distortion.

***Frequency Distortion :*** A complex (voice) voltage transmitted on a transmission line will not be attenuated equally and the received waveform will not be identical with the input waveform at the transmitting end. This variation is known as frequency distortion.

The attenuation constant is given by

$$\alpha = \sqrt{\frac{RG - \omega^2LC + \sqrt{(RG - \omega^2LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

$\alpha$ is a function of frequency and therefore the line will introduce frequency distortion.

***Delay or Phase Distortion :*** For an applied voice-voltage wave the received waveform may not be identical with the input waveform at the sending end, since some frequency components will be delayed more than those of other frequencies. This phenomenon is known as delay or phase distortion.

The phase constant is

$$\beta = \sqrt{\frac{RG - \omega^2LC + \sqrt{(RG - \omega^2LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

$\beta$ is not a constant multiplied by $\omega$ and therefore the line will introduce delay distortion.

Frequency distortion is reduced in the transmission of high quality over wire lines by the use of equalizers at the line terminals.

Delay distortion is of relatively less importance to voice and music transmission. But it can be very serious for video transmission. This can be avoided by the use of co-axial cables.

## 1.7. THE DISTORTIONLESS LINE

If a line is to have neither frequency nor delay distortion, then attenuation factor $\alpha$ and the velocity of propagation $v$ cannot be functions of frequency.

If $\quad v = \dfrac{\omega}{\beta}$

$\beta$ must be a direct function of frequency.

$$\beta = \sqrt{\frac{\omega^2LC - RG + \sqrt{(RG - \omega^2LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

For $\beta$ to be a direct function of frequency, the term

$(RG - \omega^2LC)^2 + \omega^2 (LG + CR)^2$ must be equal to $(RG + \omega^2LC)^2$

$$R^2G^2 + \omega^4L^2C^2 - 2\omega^2LCRG + \omega^2L^2G + \omega^2C^2R^2 + 2\omega^2LCRG$$

$$= R^2G^2 + \omega^4L^2C^2 + 2\omega^2LCRG$$

$$\omega^2L^2G^2 + \omega^2C^2R^2 = 2\,\omega^2LCRG$$

$$\omega^2L^2G^2 + \omega^2C^2R^2 - 2\omega^2LCRG = 0$$

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\frac{R}{L} = \frac{G}{C}$$



This is the condition for distortionless line.

$$\text{Propagation constant } \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{L\left(\frac{R}{L} + j\omega\right) C\left(\frac{G}{C} + j\omega\right)}$$

$$= \sqrt{LC}\sqrt{\left(\frac{R}{L} + j\omega\right)\left(\frac{G}{C} + j\omega\right)}$$

$$\text{But } \frac{R}{L} = \frac{G}{C}$$

$$\gamma = \sqrt{LC}\left(\frac{R}{L} + j\omega\right)$$

$$\text{Then} \quad \beta = \sqrt{\frac{\omega^2 LC - RG + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2\omega^2 LC}{2}}$$

$$\beta = \omega\sqrt{LC}$$

Velocity of propagation is $$v = \frac{\omega}{\beta}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the same velocity for all frequencies, thus eliminating delay distortion.

Attenuation factor

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

To make $\alpha$ is independent of frequency, the term $(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2$ is forced to be equal to $(RG + \omega^2 LC)^2$.

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\boxed{\frac{L}{C} = \frac{R}{G}}$$

This will make $\alpha$ and the velocity independent of frequency simultaneously. To achieve this condition, it requires a very large value of L, since G is small.

The attenuation factor $$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG + \omega^2 LC)^2}}{2}}$$

$$= \sqrt{\frac{RG - \omega^2 LC + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2RG}{2}}$$

$$\alpha = \sqrt{RG}$$

It is independent of frequency, thus eliminating frequency distortion on the line.

The characteristic impedance $Z_0$ is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{L\left(\frac{R}{L} + j\omega\right)}{C\left(\frac{G}{C} + j\omega\right)}}$$

But $\frac{R}{L} = \frac{G}{C}$ for distortionless line.

$$\therefore \quad Z_0 = \sqrt{\frac{L}{C}}$$

It is purely real and is independent of frequency.

### 1.8. TELEPHONE CABLE

In the telephone cable the wires are insulated with paper and twisted in pairs. This construction results in negligible values of inductance and conductance. Therefore $L\omega \ll R$ and $G \ll C\omega$.

$$Z = R + j\omega L \approx R$$

$$Y = G + j\omega C \approx j\omega C$$

Propagation constant $\quad \gamma = \sqrt{ZY}$

$$= \sqrt{j\omega RC}$$

$$= \sqrt{\frac{j2\omega RC}{2}}$$

But $\gamma = \alpha + j\beta$

$$\alpha + j\beta = (1 + j)\sqrt{\frac{\omega RC}{2}}$$

Equating real and imaginary parts $\quad \alpha = \sqrt{\frac{\omega RC}{2}}$



$$\beta = \sqrt{\frac{\omega RC}{2}}$$

Velocity of propagation $v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega RC}{2}}} = \sqrt{\frac{2\omega}{RC}}$

The characteristic impedance $Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R}{j\omega C}} = \sqrt{\frac{R}{\omega C}} \angle -45°$

It is found that the propagation constant $\alpha$ and velocity of propagation $v$ are functions of frequency. Thus, the higher frequencies are attenuated more and travel faster than the lower frequencies resulting in considerable frequency and delay distortion.

## 1.9. LOADING OF LINES

It is necessary to increase L/C ratio to achieve distortionless condition in a transmission line. This can be done by increasing the inductance of a transmission line. Increasing inductance by inserting inductances in series with line is termed as ***loading*** and such lines are called ***loaded lines.*** The lumped inductors, known as ***loading coils*** are placed at suitable intervals along the transmission line to increase the effective distributed inductance.

The effect of loading can be realised by comparing the unloading of a transmission line in the attenuation Vs frequency graph. Fig.1.5 shows that the loaded line offers a low attenuation when compared to the unloaded line only for limited range of frequencies.

The important aspect of loading coil design is that saturation and stray fields should be avoided. It should have a low resistance and should be in small size. In general toroidal cores are used for loading coils.

### Types of Loading

The open wire lines have more inductance of their own and so have much less distortion than cable. Therefore, the loading practice is not applicable to open wires but it is restricted to cables only. There are three types of loading in practice. They are

*(a)* Lumped loading

*(b)* Continuous loading

*(c)* Patch loading

***(a) Lumped loading :*** The inductance of a transmission line can be increased by the introduction of loading coil at uniform intervals. This is called lumped loading. It acts as a low pass filter. So, it is applicable only for a limited range of frequency. The loading coils have an internal resistance R thus, increasing the total effective inductance increases R. Further hysteresis and eddy current losses which occur in the loading coils resulting in further apparent increase in R. Therefore, there is a practical limitation on the value of inductance that can be increased for the reduction of attenuation. Thus the loading coil should be carefully designed so that it will not introduce any distortion.

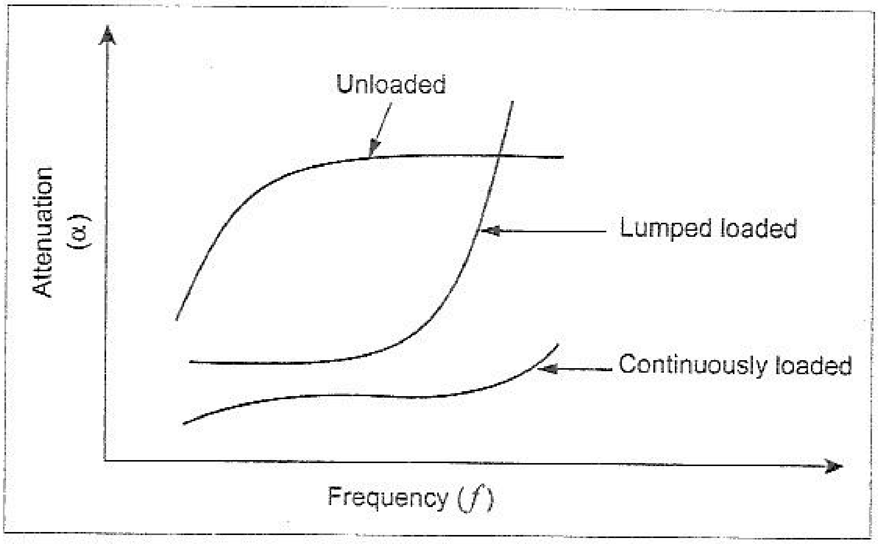



*Fig. 1.5. Comparison of loaded and unloaded cable characteristics*

***(b) Continuous loading :*** A type of iron or some other magnetic material is wound on the transmission line (cable) to increase the permeability of the surrounding medium and thereby increase the inductance. It is a quite expensive method. Further eddy current and hysteresis losses in the magnetic material increases the primary constant R. Therefore, continuous loading is used only on ocean cables where lumped loading is difficult. The advantage of continuous loading over lumped loading is that attenuation factor $\alpha$ increases uniformly with increase in frequency.

***(c) Patch loading :*** It employs sections of continuously loaded cable separated by sections of unloaded cable. The typical length for the section is normally a quarter kilometer. In this method the advantage of continuous loading is obtained and the cost is reduced considerably.

### 1.9.1. Inductance Loading of Telephone Cables

Distortionless line with distributed parameters is used to avoid the frequency and delay distortion experienced on telephone cables. It is necessary to increase the L/C to achieve distortionless condition $\frac{L}{C} = \frac{R}{G}$. Heaviside suggested that the inductance be increased and Pupin suggested that this increase in the inductance by lumped inductors spaced at intervals along the line. This use of inductance is called loading the line. The distributed loading is obtained by winding the cable with a high permeability steel tape such as permalloy in some submarine cables.

Consider an uniformly loaded cable with G = 0. Then,

$$Z = R + j\omega L$$

$$Y = j\omega C \qquad [\because G = 0]$$

$$Z = \sqrt{R^2 + (L\omega)^2} \;\underline{\Big|\, \tan^{-1}\left(\frac{L\omega}{R}\right)}$$

$$= \sqrt{R^2 + (L\omega)^2} \;\angle \frac{\pi}{2} - \tan^{-1}\frac{R}{L\omega}$$

Propagation constant $\gamma = \sqrt{ZY}$

$$= \sqrt{\sqrt{R^2 + (L\omega)^2} \;\angle \frac{\pi}{2} - \tan^{-1}\frac{R}{L\omega} \left(\omega C \angle \frac{\pi}{2}\right)}$$

$$= \sqrt{\omega C \sqrt{R^2 + (L\omega)^2} \;\angle \pi - \tan^{-1}\frac{R}{L\omega}}$$

$$= \sqrt{(\omega C)(L\omega)\sqrt{1 + \frac{R^2}{(L\omega)^2}}} \;\angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$= \omega\sqrt{LC}\;\sqrt[4]{1 + \left(\frac{R}{L\omega}\right)^2} \;\angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

Since R is small with respect to $L\omega$, the term $\left(\frac{R}{L\omega}\right)$ is neglected.

$$\therefore\ \gamma = \omega\sqrt{LC} \;\angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$\text{If}\quad \theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$\cos\theta = \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right)$$

$$= \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right)$$

$\frac{R}{2L\omega}$

For small angle,

$$\sin\theta \approx \tan\theta \approx \theta$$

so that $$\cos\theta = \frac{R}{2L\omega}$$

Similarly, $$\sin\theta = \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right) = 1$$

Propagation constant $$\gamma = \omega\sqrt{LC}\;(\cos\theta + j\sin\theta)$$

$$= \omega\sqrt{LC}\left(\frac{R}{2L\omega} + j\right)$$

$$\gamma = \frac{R\sqrt{LC}}{2L} + j\omega\sqrt{LC}$$

$$= \frac{R}{2}\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

$$\therefore \text{ Attenuation constant } \alpha = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\text{Phase-shift } \beta = \omega\sqrt{LC}$$

$$\text{Velocity of propagation } v = \frac{\omega}{\beta}$$

$$= \frac{1}{\sqrt{LC}}$$

It is noted that if G = 0 and L$\omega$ >> R, the attenuation and velocity are both independent of frequency and the loaded cable will be distortionless. Attenuation may be reduced by increasing L. Continuous (uniform) loading is expensive and achieves only a small increase in L per unit length. Lumped loading is preferred for cables.

## Campbell's Equation

An analysis for the performance of a line loaded at uniform intervals can be obtained by considering a symmetrical section of line from the centre of one loading coil to the centre of the next coil. The section of line may be replaced with an equivalent T section having symmetrical series arms as shown in Fig.1.6. The series arm of T section including loading coil is given by

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + \frac{Z_1}{2} \qquad \text{[From the fig.]}$$

where $\frac{Z_1}{2}$ is the series arm of T section.

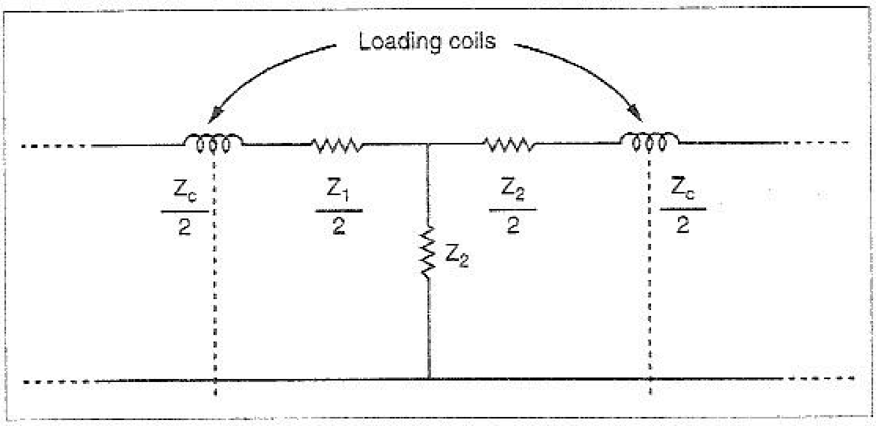



*Fig. 1.6. Equivalent T section for part of a line between two lumped loading coils*

$$\frac{Z_1}{2} = Z_0 \tanh\frac{\gamma l}{2}$$

$$\therefore \frac{Z_1'}{2} = \frac{Z_c}{2} + Z_0 \tanh\frac{\gamma l}{2}$$

Where $l$ is the distance between two loading coils.

The shunt $Z_2$ arm of the equivalent T section is

$$Z_2 = \frac{Z_o}{\sinh \gamma l}$$

For loaded T section

$$\cosh \gamma' l = 1 + \frac{Z_1'}{2 Z_2}$$

$$= 1 + \frac{\frac{Z_c}{2} + Z_o \tanh \frac{\gamma l}{2}}{\frac{Z_o}{\sinh \gamma l}}$$

$$\text{But } \tanh \frac{\gamma l}{2} = \frac{\cosh \gamma l - 1}{\sinh \gamma l}$$

Substituting this value in above equation

$$\therefore \cosh \gamma' l = 1 + \frac{\frac{Z_c}{2} + Z_o \frac{\cosh \gamma l - 1}{\sinh \gamma l}}{\frac{Z_o}{\sinh \gamma l}}$$

$$= 1 + \frac{\frac{Z_c}{2} \sinh \gamma l + Z_o (\cosh \gamma l - 1)}{Z_o}$$

$$= 1 + \frac{Z_c}{2 Z_o} \sinh \gamma l + \cosh \gamma l - 1$$

$$\cosh \gamma' l = \frac{Z_c}{2 Z_o} \sinh \gamma l + \cosh \gamma l$$

This equation is called as Campbell's equation and it is used to determine the value of $\gamma'$ of a line section consisting of partially lumped and partially distributed elements. For a cable $Z_2$ is capacitance and the cable capacitance and lumped inductances appear similar to the circuit of the low pass filter. It is found that for frequencies below cutoff, the attenuation is reduced, but the cut-off attenuation is increased (as a result of filter action). In practice, pure distortionless line is not obtained by loading, because R and L are to some extent functions of frequency. Eddy current losses are more in these coils. However, there is a major improvement in the loaded cable over the unloaded cable for a reasonable frequency range.

### 1.10. OPEN CIRCUITED AND SHORT CIRCUITED LINES

The expressions for voltage and current at the sending end of a transmission line of length '$l$' are given by

$$V_S = V_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_o}{Z_R}\sinh\sqrt{ZY}\,l\right]$$

$$I_S = I_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_o}\sinh\sqrt{ZY}\,l\right]$$

The input impedance of a transmission line is given by

$$Z_S = \frac{V_S}{I_S}$$

$$= \frac{V_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_o}{Z_R}\sinh\sqrt{ZY}\,l\right]}{I_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_o}\sinh\sqrt{ZY}\,l\right]}$$

$$= \frac{V_R}{I_R}\,\frac{Z_o}{Z_R}\,\frac{(Z_R\cosh\gamma l + Z_o\sinh\gamma l)}{(Z_o\cosh\gamma l + Z_R\sinh\gamma l)}$$

$$= Z_o\left(\frac{Z_R\cosh\gamma l + Z_o\sinh\gamma l}{Z_o\cosh\gamma l + Z_R\sinh\gamma l}\right) \qquad \left[\because Z_R = \frac{V_R}{I_R}\right]$$

$$Z_S = Z_o\left(\frac{Z_R\cosh\gamma l + Z_o\sinh\gamma l}{Z_o\cosh\gamma l + Z_R\sinh\gamma l}\right)$$

If short circuited, the receiving end impedance is zero.

$$i.e.,\quad Z_R = 0$$

$$\therefore Z_{sc} = Z_o\left(\frac{Z_o\sinh\gamma l}{Z_o\cosh\gamma l}\right)$$

Short circuited impedance

$$Z_{sc} = Z_o\tanh\gamma l$$

If open circuited, the receiving end impedance is infinite.

$$i.e.,\qquad Z_R = \infty$$

Input impedance of transmission line can be written as

$$Z_S = Z_o\left[\frac{\cosh\gamma l + \frac{Z_o}{Z_R}\sinh\gamma l}{\frac{Z_o}{Z_R}\cosh\gamma l + \sinh\gamma l}\right]$$



Applying $Z_R = \infty$

$$\text{Then} \quad Z_{oc} = Z_0 \left[ \frac{\cosh \gamma l}{\sinh \gamma l} \right]$$

The open circuited impedance

$$Z_{oc} = Z_0 \coth \gamma l$$

By multiplying open circuited impedance and short circuited impedances

$$Z_{oc} Z_{sc} = Z_0^2 \tanh \gamma l \coth \gamma l$$

$$= Z_0^2$$

The characteristic impedance is given by

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$

By dividing short circuited impedance by open circuited impedance.

$$\frac{Z_{sc}}{Z_{oc}} = \frac{Z_0 \tanh \gamma l}{Z_0 \coth \gamma l}$$

$$= \tan^2 h \, \gamma l$$

$$\tanh \gamma l = \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

$$\gamma l = \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

## 1.11. REFLECTION

When the load impedance is not equal to the characteristic impedance of transmission line, reflection takes place.

The expressions for voltage and current on the transmission line are

$$V = \frac{V_R}{2} \left[ \left(1 + \frac{Z_0}{Z_R}\right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_0}{Z_R}\right) e^{-\sqrt{ZY}\,x} \right]$$

$$I = \frac{I_R}{2} \left[ \left(1 + \frac{Z_R}{Z_0}\right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_R}{Z_0}\right) e^{-\sqrt{ZY}\,x} \right]$$

or

$$V = \frac{V_R}{2} \left[ \frac{Z_R + Z_0}{Z_R} e^{\sqrt{ZY}\,x} + \frac{Z_R - Z_0}{Z_R} e^{-\sqrt{ZY}\,x} \right]$$

$$I = \frac{I_R}{2} \left[ \frac{Z_R + Z_0}{Z_0} e^{\sqrt{ZY}\,x} - \frac{Z_R - Z_0}{Z_0} e^{-\sqrt{ZY}\,x} \right]$$

or

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R} \left[ e^{\gamma x} + \left( \frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$I = \frac{I_R (Z_R + Z_o)}{2 Z_R} \left[ e^{\gamma x} - \left( \frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$[\because \gamma = \sqrt{ZY}]$$

If the transmission line is not terminated with the characteristic impedance *i.e.*, $Z_R \neq Z_o$ (mismatch) the above expressions for voltage and current exist. It consists of two waves, one is moving in the forward (positive $x$) direction which is called incident wave and the other is moving in the opposite (negative $x$) direction which is called reflected ray. The term varying with $e^{\gamma x}$ represents a wave progressing from the sending end towards the receiving end and the amplitude decreasing with increased distance. The term varying with $e^{-\gamma x}$ represents a wave progressing from the receiving end towards the sending end, decreasing in amplitude with increased distance.

If the transmission line is terminated with characteristic impedance *i.e.*, $Z_R = Z_o$ (properly matched) then the voltage and current expressions are

$$V = V_R e^{\gamma x}$$

$$I = I_R e^{\gamma x}$$

The incident wave moves only in forward (positive $x$) direction. There is no reflected wave in the opposite direction.

### 1.11.1. Reflection Coefficient

Reflection coefficient is defined as the ratio of the reflected voltage to the incident voltage at the receiving end of the line.

$$K = \frac{\text{Reflected voltage at load}}{\text{Incident voltage at load}} = \frac{V_R}{V_S}$$

The equation for the voltage of a transmission line is

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R} \left[ e^{\gamma x} + \left( \frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R} e^{\gamma x} + \frac{V_R (Z_R - Z_o)}{2 Z_R} e^{-\gamma x}$$

The first term ($e^{\gamma x}$) represents incident wave, whereas the second term ($e^{-\gamma x}$) represents the reflected wave. The ratio of amplitude of the reflected wave voltage to the amplitude of the incident wave voltage is nothing but reflection coefficient.

$$K = \frac{\dfrac{V_R (Z_R - Z_o)}{2 Z_R}}{\dfrac{V_R (Z_R + Z_o)}{2 Z_R}} = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$\boxed{K = \frac{Z_R - Z_o}{Z_R + Z_o}}$$

It is also defined as in terms of the ratio of the reflected current to the incident current. But it is negative.

$$-K = \frac{\text{Reflected current at load}}{\text{Incident current at load}} = \frac{I_R}{I_S}$$

If the transmission line is terminated by its characteristic impedance ($Z_R = Z_o$), the reflection coefficient becomes zero.

### 1.11.2. Reflection Factor and Reflection Loss

Consider a transmission line with a voltage source $V_S$ and its impedance $Z_1$ and load impedance $Z_2$ as shown in Fig.1.7. If $Z_2$ is not equal to $Z_1$, reflection takes place. The power delivered to the load is less than that with impedance matching. Reflection results in power loss. This loss is known as reflection loss.

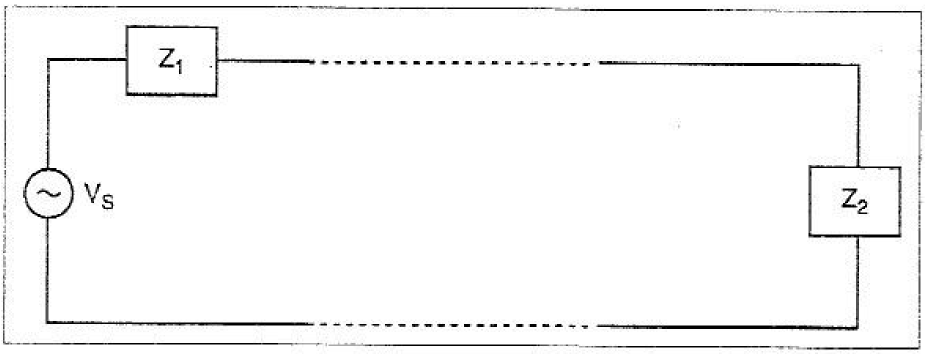



*Fig. 1.7. Transmission line with voltage source $V_S$ and impedance $Z_1$*

Image matching between the impedances $Z_1$ and $Z_2$ can be obtained by inserting an ideal transformer and a phase shifting network between $Z_1$ and $Z_2$. If $I_1$ and $I_2$ be the currents in the primary and secondary of the transformer respectively, the current ratio of the transformer is given by

$$\frac{I_2}{I_1} = \sqrt{\frac{Z_1}{Z_2}}$$

$Z_2$ may be adjusted to that of $Z_1$ by choosing the proper transformation ratio and phase angle. $Z_2$ is the image impedance of $Z_1$. The current through the source is

$$I_1 = \frac{V_S}{2\,Z_1}$$

The current flow in the secondary of the transformer under image impedance matching is

$$I_2' = I_1\sqrt{\frac{Z_1}{Z_2}} = \frac{V}{2\,Z_1}\sqrt{\frac{Z_1}{Z_2}} = \frac{V_S}{2\sqrt{Z_1 Z_2}}$$

The current in the load impedance $Z_2$ without image matching.

$$|I_2| = \frac{|V_S|}{|Z_1 + Z_2|}$$

The ratio of the current actually flowing in the load to that which might flow under matched condition is known as ***reflection factor.***

$$\left|\frac{I_2}{I_2'}\right| = \frac{\dfrac{|V_S|}{|Z_1+Z_2|}}{\dfrac{|V_S|}{|2\sqrt{Z_1 Z_2}|}}$$

$$k = \left|\frac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2}\right|$$

The reflection factor indicates the change in current in the load due to reflection at the mismatched junction.

The ***reflection loss*** is the reciprocal of the reflection factor in nepers or dB.

$$\text{Reflection loss} = \ln\frac{1}{k}$$

$$= \ln\left|\frac{Z_1+Z_2}{2\sqrt{Z_1 Z_2}}\right| \text{ nepers}$$

$$= 20\log\left|\frac{Z_1+Z_2}{2\sqrt{Z_1 Z_2}}\right| \text{ dB}$$

## 1.12. T AND $\pi$ SECTIONS EQUIVALENT TO LINES

A T section is shown in Fig.1.8 with two ports 1, 1 and 2, 2.

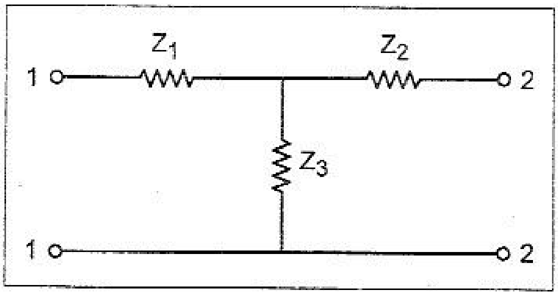



*Fig. 1.8. T section network*

Impedance measurements may be made at any port with the other port opened or shorted.

Let $Z_{1OC}$ be the impedance at port 1 when port 2 is open circuited.

$Z_{1SC}$ be the impedance at port 1 when port 2 is short circuited.

$Z_{2OC}$ be the impedance at port 2 when port 1 is open circuited.

$Z_{2SC}$ be the impedance at port 2 when port 1 is short circuited.

$$Z_{1OC} = Z_1 + Z_3$$

$$Z_{1SC} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$Z_{2OC} = Z_2 + Z_3$$

$$Z_{2SC} = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

By solving these equations, the values of $Z_1$, $Z_2$ and $Z_3$ are determined.

$$Z_{1OC} - Z_{1SC} = Z_3 - \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$= \frac{Z_3 Z_2 + Z_3^2 - Z_2 Z_3}{Z_2 + Z_3}$$

$$= \frac{Z_3^2}{Z_2 + Z_3}$$

$$= \frac{Z_3^2}{Z_{2OC}} \qquad [\because Z_2 + Z_3 = Z_{2OC}]$$

$$Z_3^2 = Z_{2OC}(Z_{1OC} - Z_{1SC})$$

$$Z_3 = \pm\sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

Taking the positive value,

$$Z_3 = \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - Z_3 \qquad [\because Z_{1OC} = Z_1 + Z_3]$$

$$= Z_{1OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_2 = Z_{2OC} - Z_3 \qquad [\because Z_{2OC} = Z_2 + Z_3]$$

$$= Z_{2OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$



$$Z_2 = Z_{2OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

# UNIT – V
# Transmission Lines – II

- SC and OC Lines
- Input Impedance Relations
- Reflection Coefficient
- VSWR
- $\lambda/4$, $\lambda$ 2, $\lambda$ /8 Lines - Impedance Transformations
- Smith Chart - Configuration and Applications,
- Single Stub Matching
- Illustrative Problems.

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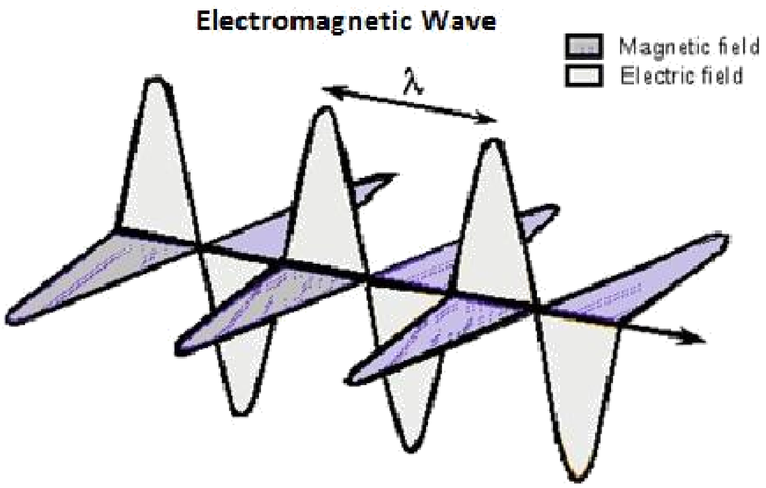
This means, more the current flows towards the surface of the conductor, it flows less towards the center, which is known as the **Skin Effect**.

## Inductance

In an AC transmission line, the current flows sinusoidally. This current induces a magnetic field perpendicular to the electric field, which also varies sinusoidally. This is well known as Faraday's law. The fields are depicted in the following figure.




This varying magnetic field induces some EMF into the conductor. Now this induced voltage or EMF flows in the opposite direction to the current flowing initially. This EMF flowing in the opposite direction is equivalently shown by a parameter known as **Inductance**, which is the property to oppose the shift in the current.

It is denoted by "**L**". The unit of measurement is "**Henry** $H$".

## Conductance

There will be a leakage current between the transmission line and the ground, and also between the phase conductors. This small amount of leakage current generally flows through the surface of the insulator. Inverse of this leakage current is termed as **Conductance**. It is denoted by "**G**".

The flow of line current is associated with inductance and the voltage difference between the two points is associated with capacitance. Inductance is associated with the magnetic field, while capacitance is associated with the electric field.

## Capacitance

The voltage difference between the **Phase conductors** gives rise to an electric field between the conductors. The two conductors are just like parallel plates and the air in between them becomes dielectric. This pattern gives rise to the capacitance effect between the conductors.

## Characteristic Impedance

If a uniform lossless transmission line is considered, for a wave travelling in one direction, the ratio of the amplitudes of voltage and current along that line, which has no reflections, is called as **Characteristic impedance**.

It is denoted by $Z_0$

$$Z_0 = \sqrt{\frac{voltage \;\; wave \;\; value}{current \;\; wave \;\; value}}$$

$$Z_0 = \sqrt{\frac{R + jwL}{G + jwC}}$$

For a lossless line, $R_0 = \sqrt{\frac{L}{C}}$

Where $L$ & $C$ are the inductance and capacitance per unit lengths.

## Impedance Matching

To achieve maximum power transfer to the load, impedance matching has to be done. To achieve this impedance matching, the following conditions are to be met.

The resistance of the load should be equal to that of the source.

$$R_L = R_S$$

The reactance of the load should be equal to that of the source but opposite in sign.

$$X_L = -X_S$$

Which means, if the source is inductive, the load should be capacitive and vice versa.

## Reflection Co-efficient

The parameter that expresses the amount of reflected energy due to impedance mismatch in a transmission line is called as **Reflection coefficient**. It is indicated by $\rho$ *rho*.

It can be defined as "the ratio of reflected voltage to the incident voltage at the load terminals".

$$\rho = \frac{reflected\ voltage}{incident\ voltage} = \frac{V_r}{V_i}\ at\ load\ terminals$$

If the impedance between the device and the transmission line don't match with each other, then the energy gets reflected. The higher the energy gets reflected, the greater will be the value of $\rho$ reflection coefficient.

## Voltage Standing Wave Ratio $VSWR$

The standing wave is formed when the incident wave gets reflected. The standing wave which is formed, contains some voltage. The magnitude of standing waves can be measured in terms of standing wave ratios.

The ratio of maximum voltage to the minimum voltage in a standing wave can be defined as Voltage Standing Wave Ratio $VSWR$. It is denoted by "$S$".

$$S = \frac{|V_{max}|}{|V_{min}|} \quad 1 \leq S \leq \infty$$

VSWR describes the voltage standing wave pattern that is present in the transmission line due to phase addition and subtraction of the incident and reflected waves.

Hence, it can also be written as

$$S = \frac{1+\rho}{1-\rho}$$

The larger the impedance mismatch, the higher will be the amplitude of the standing wave. Therefore, if the impedance is matched perfectly,

$$V_{max} : V_{min} = 1 : 1$$

Hence, the value for VSWR is unity, which means the transmission is perfect.

## Efficiency of Transmission Lines

The efficiency of transmission lines is defined as the ratio of the output power to the input power.

$$\%\ efficiency\ of\ transmission\ line\ \eta = \frac{Power\ delivered\ at\ reception}{Power\ sent\ from\ the\ transmission\ end} \times 100$$

## Voltage Regulation

Voltage regulation is defined as the change in the magnitude of the voltage between the sending and receiving ends of the transmission line.

$$\%\ voltage\ regulation = \frac{sending\ end\ voltage - receiving\ end\ voltage}{sending\ end\ voltage} \times 100$$

## Losses due to Impedance Mismatch

The transmission line, if not terminated with a matched load, occurs in losses. These losses are many types such as attenuation loss, reflection loss, transmission loss, return loss, insertion loss, etc.

## Attenuation Loss

The loss that occurs due to the absorption of the signal in the transmission line is termed as Attenuation loss, which is represented as

$$Attenuation\ loss(dB) = 10\ log_{10}\left[\frac{E_i - E_r}{E_t}\right]$$

Where

- $E_i$ = the input energy
- $E_r$ = the reflected energy from the load to the input
- $E_t$ = the transmitted energy to the load

## Reflection Loss

The loss that occurs due to the reflection of the signal due to impedance mismatch of the transmission line is termed as Reflection loss, which is represented as

$$Reflection\ loss(dB) = 10\ log_{10}\left[\frac{E_i}{E_i - E_r}\right]$$

Where

- $E_i$ = the input energy
- $E_r$ = the reflected energy from the load

## Transmission Loss

The loss that occurs while transmission through the transmission line is termed as Transmission loss, which is represented as

$$Transmission\ loss(dB) = 10\ log_{10}\ \frac{E_i}{E_t}$$

Where

- $E_i$ = the input energy
- $E_t$ = the transmitted energy

## Return Loss

The measure of the power reflected by the transmission line is termed as Return loss, which is represented as

$$Return\ loss(dB) = 10\ log_{10}\ \frac{E_i}{E_r}$$

Where

- $E_i$ = the input energy
- $E_r$ = the reflected energy

## Insertion Loss

The loss that occurs due to the energy transfer using a transmission line compared to energy transfer without a transmission line is termed as Insertion loss, which is represented as

$$Insertion\ loss(dB) = 10\ log_{10}\ \frac{E_1}{E_2}$$

Where

- $E_1$ = the energy received by the load when directly connected to the source, without a transmission line.
- $E_2$ = the energy received by the load when the transmission line is connected between the load and the source.

## Stub Matching

If the load impedance mismatches the source impedance, a method called "Stub Matching" is sometimes used to achieve matching.

The process of connecting the sections of open or short circuit lines called **stubs** in the shunt with the main line at some point or points, can be termed as **Stub Matching**.

At higher microwave frequencies, basically two stub matching techniques are employed.

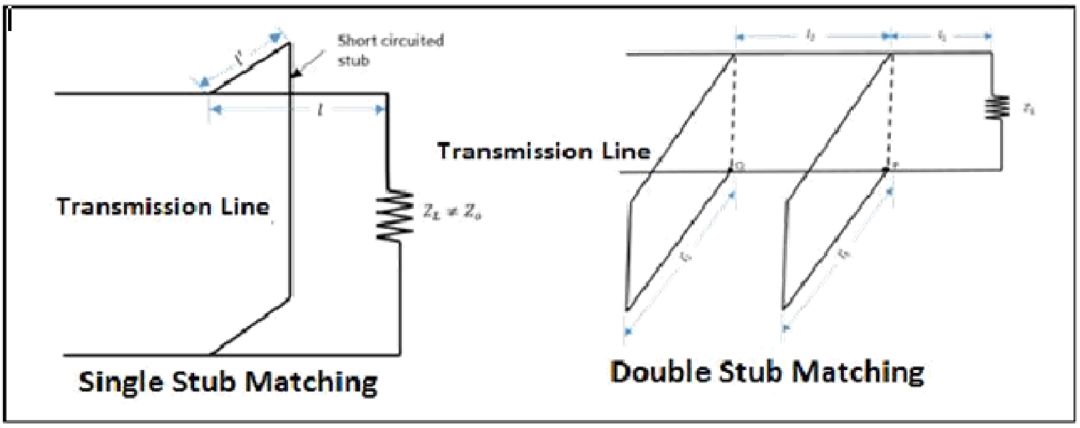
## Single Stub Matching

In Single stub matching, a stub of certain fixed length is placed at some distance from the load. It is used only for a fixed frequency, because for any change in frequency, the location of the stub has to be changed, which is not done. This method is not suitable for coaxial lines.

## Double Stub Matching

In double stud matching, two stubs of variable length are fixed at certain positions. As the load changes, only the lengths of the stubs are adjusted to achieve matching. This is widely used in laboratory practice as a single frequency matching device.

The following figures show how the stub matchings look.




Single Stub Matching

Double Stub Matching

# Transmission Lines – Smith Chart & Impedance Matching (Intensive Reading)

**1 The Smith Chart**

Transmission line calculations − such as the determination of input impedance using equation (4.30) and the reflection coefficient or load impedance from equation (4.32) − often involves tedious manipulation of complex numbers. This tedium can be alleviated using a graphical method of solution. The best known and most widely used graphical chart is the Smith chart. The Smith chart is a circular plot with a lot of interlaced circles on it. When used correctly, impedance matching can be performed without any computation. The only effort required is the reading and following of values along the circles.

The Smith chart is a polar plot of the complex reflection coefficient, or equivalently, a graphical plot of normalized resistance and reactance functions in the reflection-coefficient plane. To understand how the Smith chart for a lossless transmission line is constructed, examine the voltage reflection coefficient of the load impedance defined by

$$\Gamma_L = \frac{V_{\text{refl}}}{V_{\text{inc}}} = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{\text{re}} + j\Gamma_{\text{im}}, \tag{1}$$

where $\Gamma_{\text{re}}$ and $\Gamma_{\text{im}}$ are the real and imaginary parts of the complex reflection coefficient $\Gamma_L$. The characteristic impedance $Z_0$ is often a constant and a real industry normalized value, such as 50 Ω, 75 Ω, 100 Ω, and 600 Ω. We can then define the normalised load impedance by

$$z_L = Z_L / Z_0 = (R + jX)/Z_0 = r + jx. \tag{2}$$

With this simplification, we can rewrite the reflection coefficient formula in (1) as

$$\Gamma_L = \Gamma_{\text{re}} + j\Gamma_{\text{im}} = \frac{(Z_L - Z_0)/Z_0}{(Z_L + Z_0)/Z_0} = \frac{z_L - 1}{z_L + 1}. \tag{3}$$

The inverse relation of (3) is

$$z_L = \frac{1+\Gamma_L}{1-\Gamma_L} = \frac{1+|\Gamma_L|e^{j\theta}}{1-|\Gamma_L|e^{j\theta}} \tag{4}$$

or

$$r + jx = \frac{(1+\Gamma_{\text{re}}) + j\Gamma_{\text{im}}}{(1-\Gamma_{\text{re}}) - j\Gamma_{\text{im}}}. \tag{5}$$

Multiplying both the numerator and the denominator of (5) by the complex conjugate of the denominator and separating the real and imaginary parts, we obtain

$$r = \frac{1-\Gamma_{\text{re}}^2 - \Gamma_{\text{im}}^2}{(1-\Gamma_{\text{re}})^2 + \Gamma_{\text{im}}^2} \tag{6}$$

and

$$x = \frac{2\Gamma_{\text{im}}^2}{(1-\Gamma_{\text{re}})^2 + \Gamma_{\text{im}}^2}. \tag{7}$$

Equation (6) can be rearranged as

$$\left(\Gamma_{\text{re}} - \frac{r}{1+r}\right)^2 + \Gamma_{\text{im}}^2 = \left(\frac{1}{1+r}\right)^2. \tag{8}$$

This equation is a relationship in the form of a parametric equation $(x-a)^2+(y-b)^2=R^2$ in the complex plane $(\Gamma_{re},\Gamma_{im})$ of a circle centred at the coordinates $\left(\frac{r}{r+1},0\right)$ and having a radius of $\frac{1}{r+1}$. Different values of $r$ yield circles of different radii with centres at different positions on the $\Gamma_{re}$-axis. The following properties of the $r$-circles are noted:

- The centres of all $r$-circles lie on the $\Gamma_{re}$-axis.
- The circle where there is no resistance ($r = 0$) is the largest. It is centred at the origin and has a radius of 1.
- The $r$-circles become progressively smaller as $r$ increases from 0 to ∞, ending at the $(\Gamma_{re}=1,\Gamma_{im}=0)$ point for an open circuit.
- All the $r$-circles pass through the point $(\Gamma_{re}=1,\Gamma_{im}=0)$.

See Figure 1 for further details.

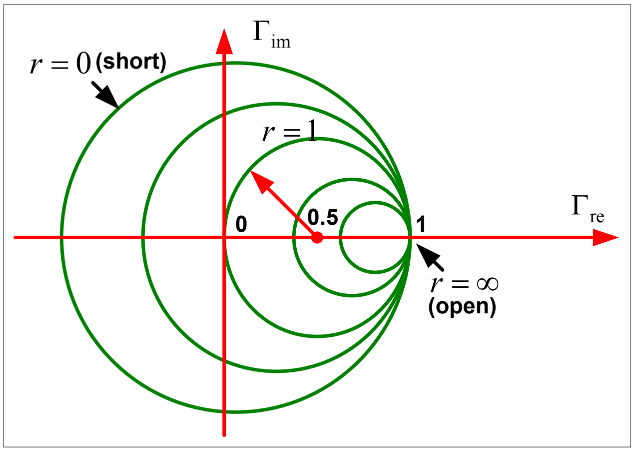



Figure 1: The $r$-circles in the complex plane $(\Gamma_{re},\Gamma_{im})$.

Similarly, (7) can be rearranged as

$$(\Gamma_{re}-1)^2+\left(\Gamma_{im}-\frac{1}{x}\right)^2=\left(\frac{1}{x}\right)^2. \qquad (9)$$

Again, (9) is a parametric equation of the type $(x-a)^2+(y-b)^2=R^2$ in the complex plane $(\Gamma_r,\Gamma_i)$ of a circle centred at the coordinates $\left(1,\frac{1}{x}\right)$ and having a radius of $\frac{1}{|x|}$. Different values of $x$ yield circles of different radii with centres at different positions on the $\Gamma_{re}=1$ line. The following properties of the $x$-circles are noted:

- The centres of all $x$-circles lie on the $\Gamma_{re}=1$ line; those for $x>0$ (inductive reactance) lie above the $\Gamma_{re}$-axis, and those for $x<0$ lie below the $\Gamma_{re}$-axis.
- The $x = 0$ circle becomes the $\Gamma_{re}$-axis.
- The $x$-circles become progressively smaller as $|x|$ increases from 0 to ∞, ending at the $(\Gamma_{re}=1,\Gamma_{im}=0)$ point for an open circuit.
- All the $x$-circles pass through the point $(\Gamma_{re}=1,\Gamma_{im}=0)$.

See Figure 2 for further details.

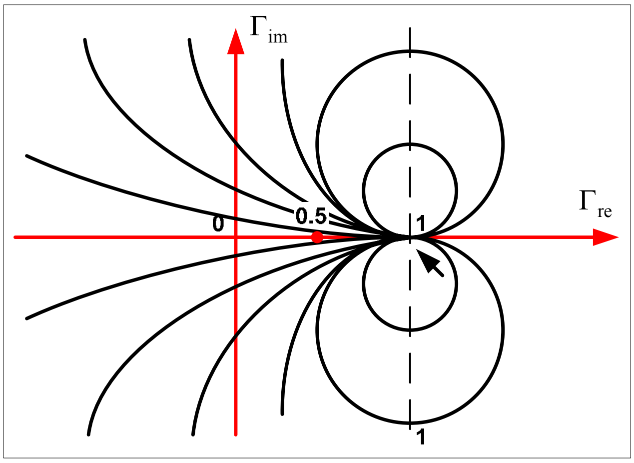



Figure 2: The $x$-circles in the complex plane $(\Gamma_{re}, \Gamma_{im})$.

To complete the Smith chart, the two circles' families are superimposed. The Smith chart therefore becomes a chart of $r$- and $x$-circles in the $(\Gamma_{re}, \Gamma_{im})$ -plane for $|\Gamma| \le 1$ . The intersection of an $r$-circle and an $x$-circle defines a point which represents a normalized load impedance $z_L = r + jx$ . The actual load impedance is $Z_L = Z_0 z_L = Z_0 (r + jx)$ . As an illustration, the impedance $Z_L = 85 + j30$ in a $Z_0 = 50\,\Omega$ -system is represented by the point $P$ in Figure 3. Here $z_L = 1.7 + j0.6$ at the intersection of the $r = 1.7$ and the $x = 0.6$ circles. Values for $\Gamma_{re}$ and $\Gamma_{im}$ may then be obtained from the projections onto the horizontal and vertical axes (see Figure 4). These are approximately given by $\Gamma_{re} \approx 0.3$ and $\Gamma_{im} \approx 0.16$ . Point $P_{sc}$ at $(\Gamma_{re} = -1, \Gamma_{im} = 0)$ corresponds to $r = 0$ and $x = 0$ and therefore represents a short-circuit. $P_{oc}$ at $(\Gamma_{re} = 1, \Gamma_{im} = 0)$ corresponds to an infinite impedance therefore represents an open circuit.

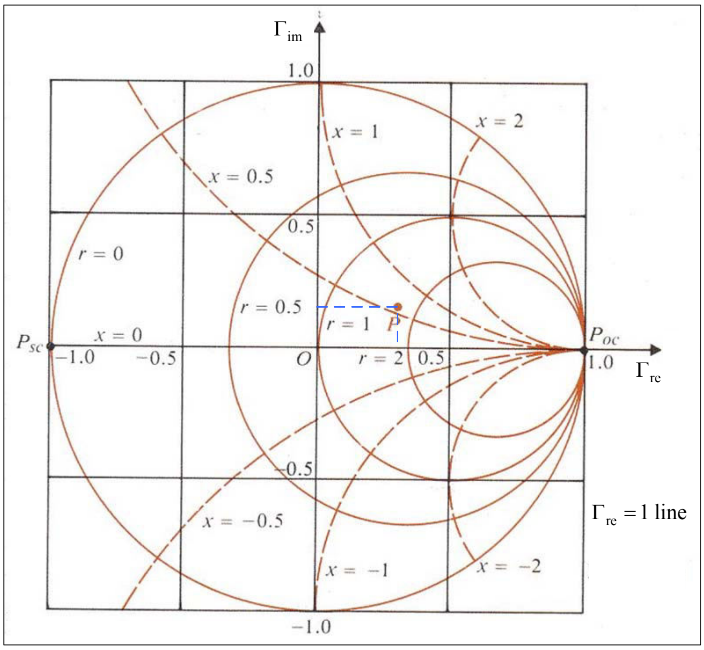



Figure 3: Smith chart with rectangular coordinates.

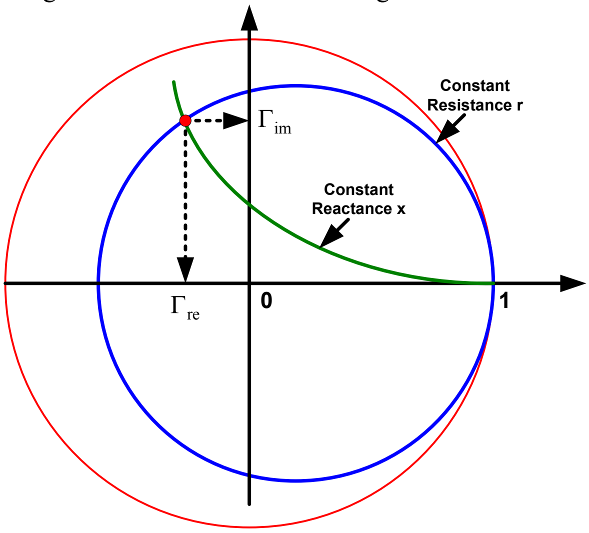



Figure 4: Direct extraction of the reflection coefficient $\Gamma = \Gamma_{re} + j\Gamma_{im}$ along the horizontal and vertical axes.

Instead of having a Smith chart marked with $\Gamma_{re}$ and $\Gamma_{im}$ marked in rectangular coordinates, the same chart can be marked in polar coordinates, so that every point in the Γ-plane is specified by a magnitude $|\Gamma|$ and a phase angle $\theta$. This is illustrated in Figure 5, where several $|\Gamma|$-circles are shown in dashed lines and some $\theta$-angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$-circles are normally not shown on commercially available Smith charts, but once the point representing a certain $z_L = r + jx$ is located, it is simply a matter of drawing a circle centred at the origin through the point. The ratio of the distance to the point and the radius to the edge of the chart is equal to the magnitude of $|\Gamma|$ of the load reflection coefficient, and the angle that a line to that point makes with the real axis represents $\theta$. If, for example the point $z_L = 1.7 + j0.6$ is marked on the Smith chart at point *P*, we find that $|\Gamma_L| = 1/3$ and $\theta = 28°$.

Each $|\Gamma|$-circle intersects the real axis at two points. In Figure 5 we designate the point on the positive real axis as $P_M$ and on the negative real axis as $P_m$. Since $x = 0$ along the real axis, both these points represent situations of a purely resistive load, $Z_L = R_L$. Obviously, $R_L > Z_0$ at $P_M$ where $r > 1$, and $R_L < Z_0$ at $P_m$ where $r < 1$. Since $S = R_L / Z_0$ for $R_L > Z_0$, the value of the *r*-circle passing through the point $P_M$ is numerically equal to the standing wave ratio. For the example where $z_L = 1.7 + j0.6$, we find that $r = 2$ at $P_M$, so that $S = r = 2$.

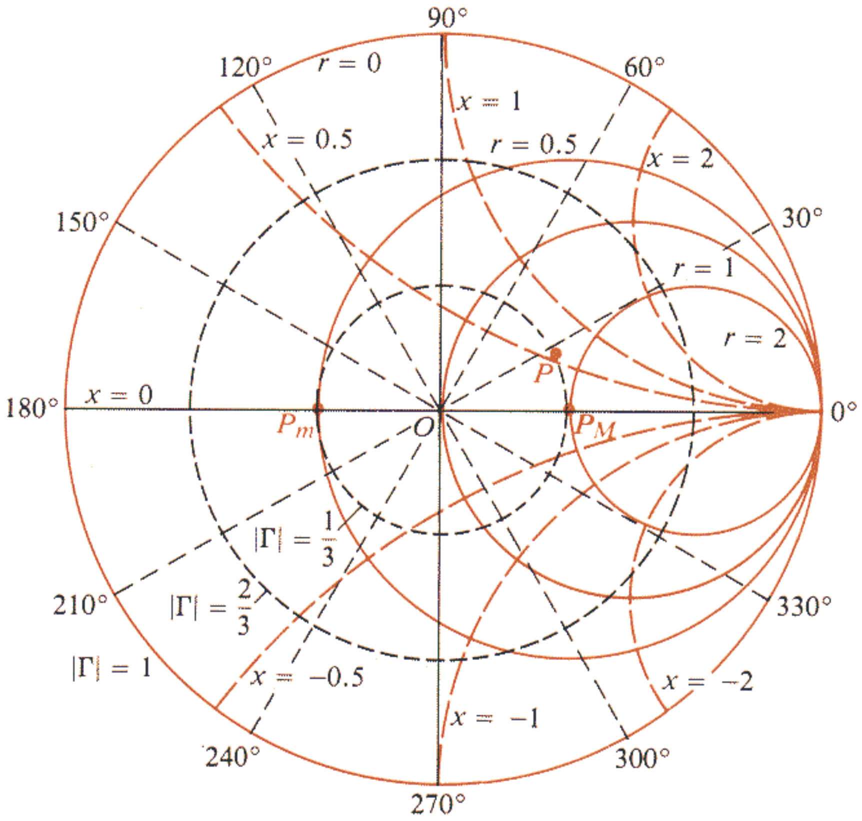



Figure 5: Smith chart in polar coordinates.

**Example 1:**

Consider a characteristic impedance of 50 Ω with the following impedances:

| | | | |
|---|---|---|---|
| $Z_1 = 100 + j50\ \Omega$ | $Z_2 = 75 - j100\ \Omega$ | $Z_3 = j200\ \Omega$ | $Z_4 = 150\ \Omega$ |
| $Z_5 = \infty$ (an open circuit) | $Z_6 = 0$ (a short circuit) | $Z_7 = 50\ \Omega$ | $Z_8 = 184 - j900\ \Omega$ |

The normalized impedances shown below are plotted in Figure 6.

| | | | |
|---|---|---|---|
| $z_1 = 2 + j$ | $z_2 = 1.5 - j2$ | $z_3 = j4$ | $z_4 = 3$ |
| $z_5 = \infty$ | $z_6 = 0$ | $z_7 = 1$ | $z_8 = 3.68 - j18$ |

It is also possible to directly extract the reflection coefficient Γ on the Smith chart of Figure 6. Once the impedance point is plotted (the intersection point of a constant resistance circle and of a constant reactance circle), simply read the rectangular coordinates projection on the horizontal and vertical axis. This will give $\Gamma_{re}$, the real part of the reflection coefficient, and $\Gamma_{im}$, the imaginary part of the reflection coefficient. Alternatively, the reflection coefficient may be obtained in polar form by using the scales provided on the commercial Smith chart.

| | | | |
|---|---|---|---|
| $\Gamma_1 = 0.4 + 0.2\,j$ <br> $= 0.45 \angle 27°$ | $\Gamma_2 = 0.51 - 0.4\,j$ <br> $= 0.65 \angle{-38°}$ | $\Gamma_3 = 0.875 + 0.48j$ <br> $= 0.998 \angle 29°$ | $\Gamma_4 = 0.5$ <br> $= 0.5 \angle 0°$ |
| $\Gamma_5 = 1$ <br> $= 1 \angle 0°$ | $\Gamma_6 = -1$ <br> $= 1 \angle 180°$ | $\Gamma_7 = 0$ <br> $= 0$ | $\Gamma_8 = 0.96 - 0.1\,j$ <br> $= 0.97 \angle{-6°}$ |

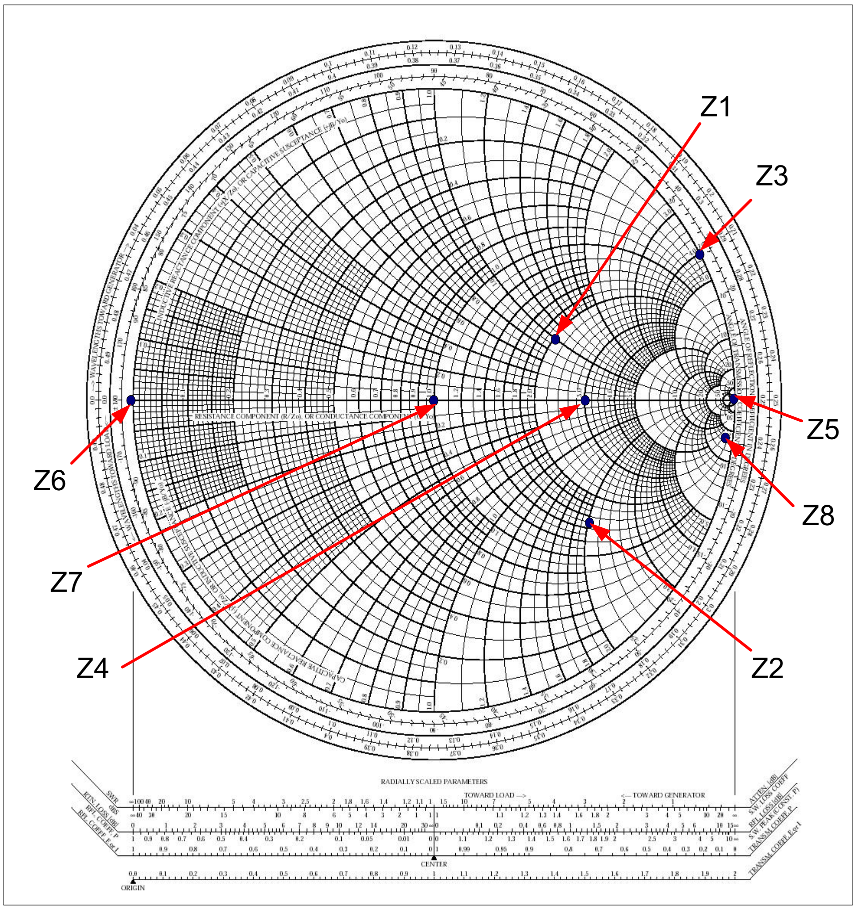



Figure 6: Points plotted on the Smith chart for Example 1.

The Smith chart is constructed by considering impedance (resistance and reactance). It can be used to analyse these parameters in both the series and parallel worlds. Adding elements in a series is straightforward. New elements can be added and their effects determined by simply moving along the circle to their respective values. However, summing elements in parallel is another matter, where admittances should be added.

We know that, by definition, $Y = 1/Z$ and $Z = 1/Y$. The admittance is expressed in mhos or $\Omega^{-1}$ or alternatively in Siemens or S. Also, as $Z$ is complex, $Y$ must also be complex. Therefore

$$Y = G + jB\,, \tag{10}$$

where $G$ is called the conductance and $B$ the susceptance of the element. When working with admittance, the first thing that we must do is normalize $y = Y/Y_0$. This results in $y = g + jb = 1/z$ . So, what happens to the reflection coefficient? We note that

$$\Gamma = \frac{z-1}{z+1} = \frac{(z-1)/z}{(z+1)/z} = \frac{1-y}{1+y} = -\left(\frac{y-1}{1+y}\right). \tag{11}$$

Thus, for a specific normalized impedance, say $z_1 = 1.7 + j0.6$, we can find the corresponding reflection coefficient as $\Gamma_1 = 0.33\angle 28°$. From (11), it then follows that the reflection coefficient for a normalized admittance of $y_2 = 1.7 + j0.6$ will be $\Gamma_2 = -\Gamma_1 = 0.33\angle(28° + 180°)$.

This also implies that for a specific normalized impedance $z$, we can find $y = 1/z$ by rotating through an angle of 180° around the centre of the Smith chart on a constant radius (see Figure 7).

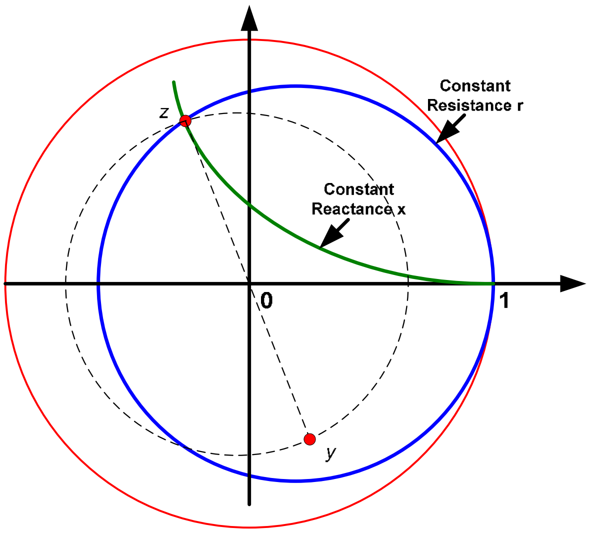



Figure 7: Results of the 180° rotation

Note that while $z$ and $y = 1/z$ represent the same component, the new point has a different position on the Smith chart and a different reflection value. This is due to the fact that the plot for $z$ is an impedance plot, but for $y$ it is an admittance plot. When solving problems where elements in series and in parallel are mixed together, we can use the same Smith chart by simply performing rotations where conversions from $z$ to $y$ or $y$ to $z$ are required.

**2 Smith Charts and transmission line circuits**

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient at the load. The question is: what happens when we connect the load to a length of transmission line as in Figure 8.

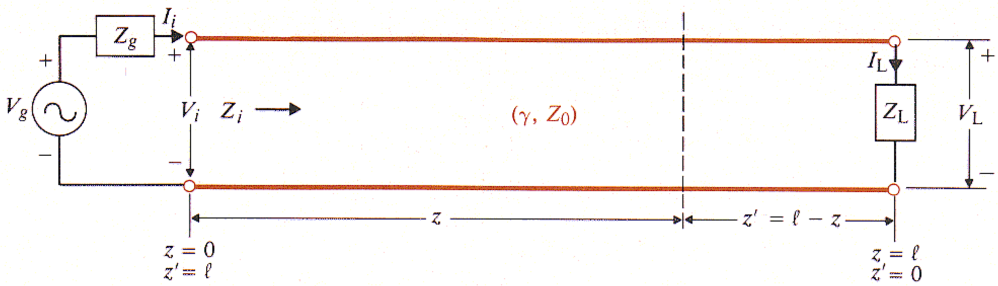



Figure 8: Finite transmission line terminated with load impedance $Z_L$.

On a lossless transmission line with $k = \beta$, the input impedance at a distance $z'$ from the load is given by

$$Z_i = \frac{V(z')}{I(z')} = Z_0 \frac{1 + \Gamma_L \, e^{-j2\beta z'}}{1 - \Gamma_L \, e^{-j2\beta z'}} . \tag{12}$$

The normalised impedance is then

$$z_i = \frac{Z_i(z')}{Z_0} = \frac{1 + \Gamma_L \, e^{-j2\beta z'}}{1 - \Gamma_L \, e^{-j2\beta z'}} = \frac{1 + \Gamma_i}{1 - \Gamma_i} . \tag{13}$$

Consequently, the reflection coefficient seen looking into the lossless transmission line of length $z'$ is given by

$$\Gamma_i = \Gamma_L \, e^{-j2\beta z'} = \left|\Gamma_L\right| e^{j\theta} \, e^{-j2\beta z'} \tag{14}$$

This implies that as we move along the transmission line towards the generator, the magnitude of the reflection coefficient does not change; the angle only changes from a value of $\theta$ at the load to a value of $(\theta - 2\beta z')$ at a distance $z'$ from the load. On the Smith chart, we are therefore rotating on a constant $|\Gamma|$ circle. One full rotation around the Smith chart requires that $2\beta z' = 2\pi$ , so that $z' = \pi/\beta = \lambda/2$ where $\lambda$ is the wavelength on the transmission line.

Two additional scales in $\Delta z'/\lambda$ are usually provided along the perimeter of the $|\Gamma| = 1$ circle for easy reading of the phase change $2\beta\Delta z'$ due to a change in line length $\Delta z'$. The outer scale is marked in "wavelengths towards generator" in the clockwise direction (increasing $z'$) and "wavelengths towards load" in the counter-clockwise direction (decreasing $z'$). Figure 9 shows a typical commercially available Smith chart.

Each $|\Gamma|$-circle intersects the real axis at two points. Refer to Figure 5. We designate the point on the positive real axis as $P_M$ and on the negative real axis as $P_m$. Since $x = 0$ along the real axis, both these points represent situations of a purely resistive input impedance, $Z_i = R_i + j0$. Obviously, $R_i > Z_0$ at $P_M$ where $r > 1$, and $R_i < Z_0$ at $P_m$ where $r < 1$. At the point $P_M$ we find that $Z_i = R_i = SZ_0$ , while $Z_i = R_i = Z_0 / S$ at $P_m$. The point $P_M$ on an impedance chart corresponds to the positions of a voltage maximum (and current minimum) on the transmission line, while $P_m$ represents a voltage minimum (and current maximum). Given an arbitrary normalised impedance $z$, the value of the $r$-circle passing through the point $P_M$ is numerically equal to the standing wave ratio. For the example, if $z = 1.7 + j0.6$, we find that $r = 2$ at $P_M$, so that $S = r = 2$ .

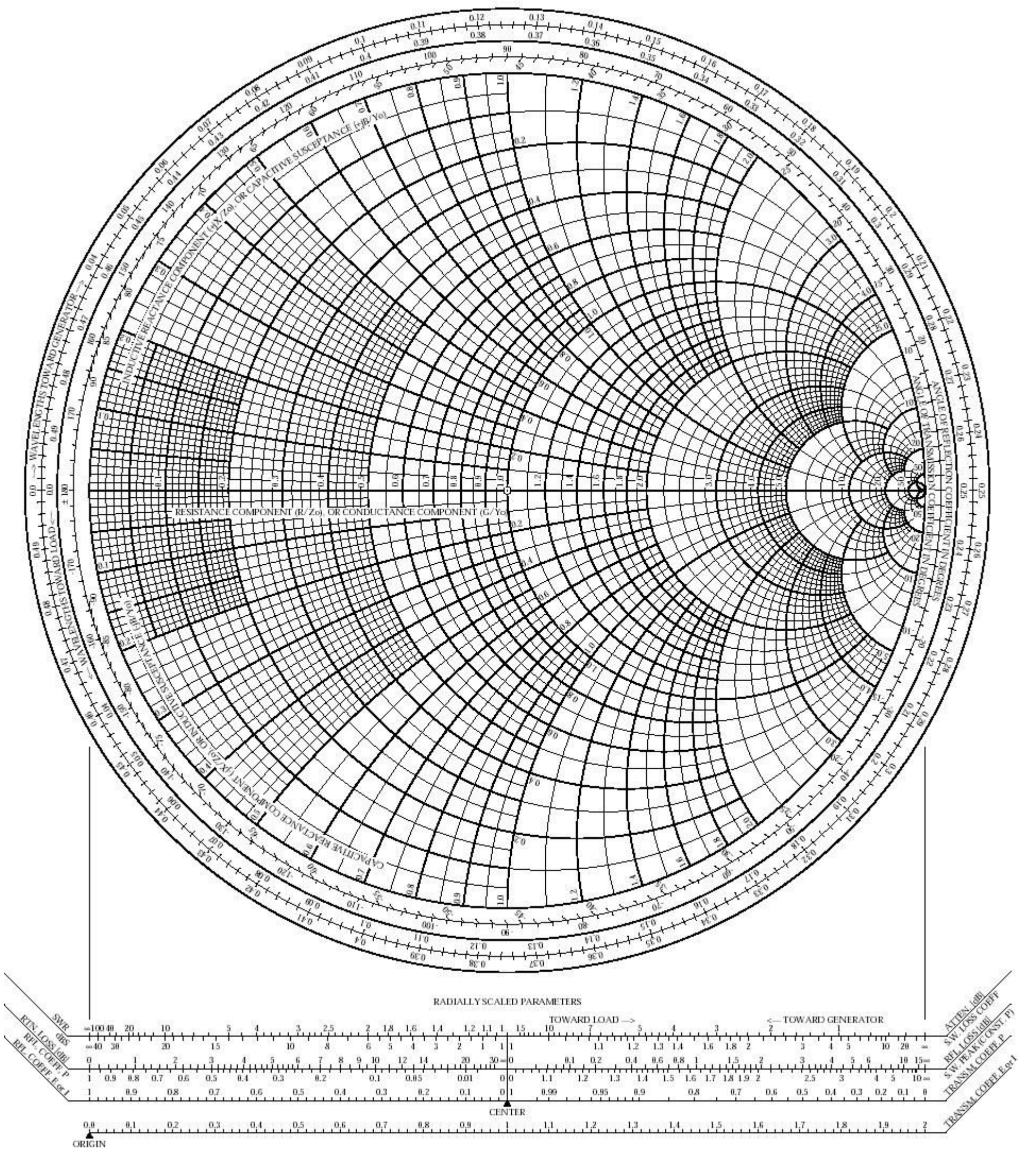



Figure 9: The Smith chart.

**Example 2:**

Use the Smith chart to find the impedance of a short-circuited section of a lossless 50 Ω co-axial transmission line that is 100 mm long. The transmission line has a dielectric of relative permittivity $\varepsilon_r = 9$ between the inner and outer conductor, and the frequency under consideration is 100 MHz.

For the transmission line, we find that $\beta = \omega\sqrt{\mu_0 \varepsilon_0 \varepsilon_r} = 6.2875$ rad/m and $\lambda = 2\pi/\beta = 0.9993 \approx 1$ m. The transmission line of length $z' = 100$ mm is therefore $z'/\lambda = 0.1$ wavelengths long.

- Since $z_L = 0$, enter the Smith chart at a point $P_{sc}$.
- Move along the perimeter of the chart ( $|\Gamma| = 1$ ) by 0.1 "wavelengths towards the generator" in a clockwise direction to point $P_1$.
- At $P_1$, read $r = 0$ and $x \approx 0.725$, or $z_i = j0.725$. Then $Z_i = j0.725 \times 50 = j36.3\,\Omega$.

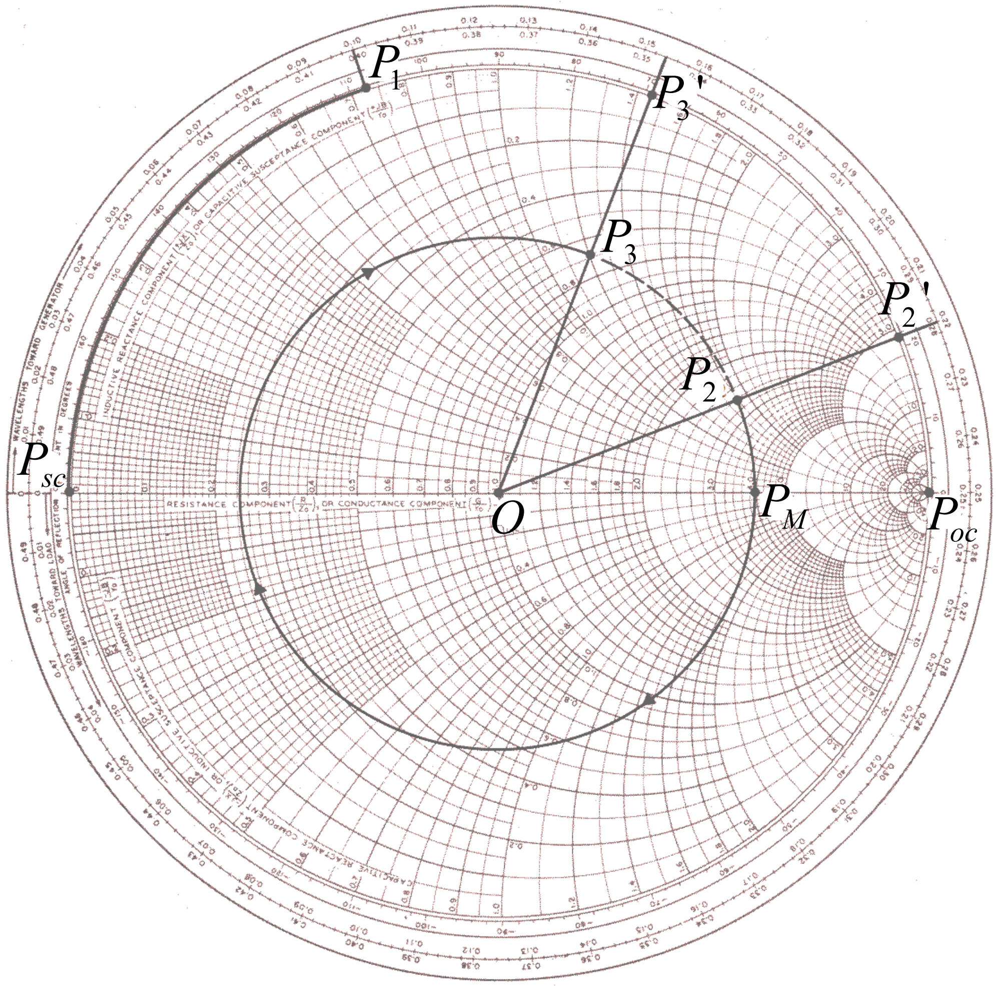



Figure 10: Smith chart calculations for Example 2 and Example 3.

**Example 3:** A lossless transmission line of length 0.434λ and characteristic impedance 100 Ω is terminated in an impedance 260 + $j$180 Ω. Find the voltage reflection coefficient, the standing-wave ratio, the input impedance, and the location of a voltage maximum on the line.

Given $z' = 0.434\lambda$, $Z_0 = 100\,\Omega$ and $Z_L = 260 + j180\,\Omega$. Then

- Enter the Smith chart at $z_L = Z_L / Z_0 = 2.6 + j1.8$ shown as point $P_2$ in Figure 10.
- With the centre at the origin, draw a circle of radius $\overline{OP_2} = |\Gamma_L| = 0.6$.
- Draw the straight line $OP_2$ and extend it to $P_2'$ on the periphery. Read 0.220 on "wavelengths towards generator" scale. The phase angle θ of the load reflection may either be read directly from the Smith chart as 21° on the "Angle of Reflection Coefficient" scale. Therefore $\Gamma_L = 0.6\,e^{j21\pi/180} = 0.6\,e^{j0.12\pi}$.
- The $|\Gamma| = 0.6$ circle intersects the positive real axis $OP_{sc}$ at $r = S = 4$. Therefore the voltage standing-wave ratio is 4.
- The find the input impedance, move $P_2'$ at 0.220 by a total of 0.434 "wavelengths toward the generator" first to 0.500 (same as 0.000) and then further to 0.434–(0.500–0.220)=0.154 to $P_3'$.

- Join $O$ and $P_3'$ by a straight line which intersects the $|\Gamma| = 0.6$ circle at $P_3$. Here $r = 0.69$ and $x = 1.2$, or $z_i = 0.69 + j1.2$. Then $Z_i = (0.69 + j1.2) \times 100 = 69 + j120\,\Omega$.
- In going from $P_2$ to $P_3$, the $|\Gamma| = 0.6$ circle intersects the positive real axis at $P_M$ where there is a voltage maximum. Thus the voltage maximum appears at 0.250–0.220=0.030 wavelengths from the load.

**3 Transmission line impedance matching.**

Transmission lines are often used for the transmission of power and information. For RF power transmission, it is highly desirable that as much power as possible is transmitted from the generator to the load and that as little power as possible is lost on the line itself. This will require that the load be matched to the characteristic impedance of the line, so that the standing wave ratio on the line is as close to unity as possible. For information transmission it is essential that the lines be matched, because mismatched loads and junctions will result in echoes that distort the information-carrying signal.

**Impedance matching by quarter-wave transformer**

For a lossless transmission line of length $l$, characteristic impedance of $Z_0 = R_0$ and terminated in a load impedance $Z_L$, the input impedance is given by

$$\begin{aligned} Z_i &= R_0 \frac{Z_L + jR_0 \tan\beta l}{R_0 + jZ_L \tan\beta l} \\ &= R_0 \frac{Z_L + jR_0 \tan(2\pi\, l/\lambda)}{R_0 + jZ_L \tan(2\pi\, l/\lambda)}. \end{aligned} \tag{15}$$

If the transmission line has a length of $l = \lambda/4$, this reduces to

$$\begin{aligned} Z_i &= R_0 \frac{Z_L + jR_0 \tan(\pi/2)}{R_0 + jZ_L \tan(\pi/2)} \\ &= R_0 \frac{Z_L/\tan(\pi/2) + jR_0}{R_0/\tan(\pi/2) + jZ_L} \\ &= R_0 \frac{0 + jR_0}{0 + jZ_L} \\ &= \frac{(R_0)^2}{Z_L}. \end{aligned} \tag{16}$$

This presents us with a simple way of matching a resistive load $Z_L = R_L$ to a real-valued input impedance of $Z_i = R_i$: insert a quarter-wave transformer with characteristic impedance $R_0$. From (16), we have $R_i = (R_0)^2 / R_L$, or

$$R_0 = \sqrt{R_i\, R_L}\,. \tag{17}$$

Note that the length of the transmission line has to be chosen to be equal to a quarter of a <u>transmission line wavelength</u> at the frequency where matching is desired. This matching method is therefore frequency sensitive, since the transmission line section will no longer be a quarter of a wavelength long at other frequencies. Also note that since the load is usually matched to a purely real impedance $Z_i = R_i$, this method of impedance matching can only be applied to <u>resistive loads</u> $Z_L = R_L$, and is not useful for matching complex load impedances to a lossless (or low-loss) transmission line.

**Example 4**

A signal generator has an internal impedance of 50 Ω. It needs to feed equal power through a lossless 50 Ω transmission line with a phase velocity of 0.5$c$ to two separate resistive loads of

64 Ω and 25 Ω at a frequency of 10 MHz. Quarter-wave transformers are used to match the loads to the 50 Ω line, as shown in Figure 11.
(a) Determine the required characteristic impedances and physical lengths of the quarter-wavelength lines.
(b) Find the standing-wave ratios on the matching line sections.

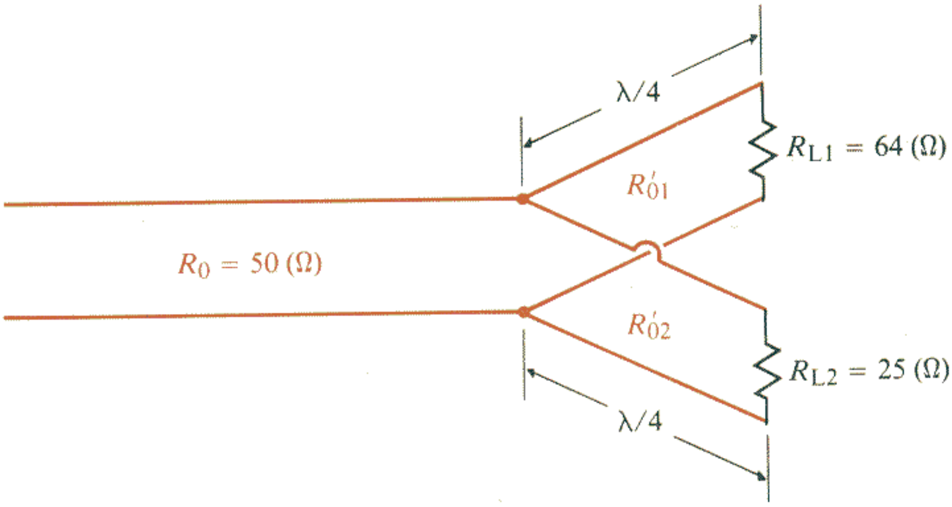



Figure 11: Impedance matching by quarter-wave transformers (Example 4).

(a) To feed equal power to the two loads, the input resistance at the junction with the main line looking toward each load must be

$$R_{i1} = 2R_0 = 100\,\Omega \qquad \text{and} \qquad R_{i2} = 2R_0 = 100\,\Omega$$

Therefore

$$R'_{01} = \sqrt{R_{i1}\,R_{L1}} = 80\,\Omega$$

$$R'_{02} = \sqrt{R_{i2}\,R_{L2}} = 50\,\Omega$$

Assume that the matching sections use the same dielectric as the main line. We know that

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{2}.$$

We can therefore deduce that it uses a dielectric with a relative permittivity of $\varepsilon_r = 4$.

$$\lambda = \frac{u_p}{f} = \frac{2\pi}{k} = 15 \text{ m.}$$

The length of each transmission line section is therefore $l = \lambda/4 = 3.75\text{ m}$.

(b) Under matched conditions, there are no standing waves on the main transmission line, i.e. $S = 1$. The standing wave ratios on the two matching line sections are as follows:
Matching section No. 1:

$$\Gamma_{L1} = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64-80}{64+80} = -0.11$$

$$S_1 = \frac{1+|\Gamma_{L1}|}{1-|\Gamma_{L1}|} = \frac{1+0.11}{1-0.11} = 1.25$$

Matching section No. 2:

$$\Gamma_{L2} = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33$$

$$S_2 = \frac{1 + |\Gamma_{L2}|}{1 - |\Gamma_{L2}|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$

**Single stub matching**

In matching of impedances, we are only allowed to use reactive components (i.e. equivalent to inductors and capacitors – no resistors). Recall that for short-circuited and open-circuited lossless transmission line sections of length *l*, the input impedance was given by

$$Z_{i,s} = jZ_0 \tan\beta l = jZ_0 \tan(2\pi\, l/\lambda)\,, \tag{18}$$

and

$$Z_{i,o} = -jZ_0 \cot\beta l = -jZ_0 \cot(2\pi\, l/\lambda)\,, \tag{19}$$

where $Z_0 = R_0$ is purely real. The impedances in (18) and (19) are purely reactive (imaginary), and therefore these transmission line sections act as inductors or capacitors, depending on the line length. We are going to make use of these elements (called transmission line *stubs*) to design matching circuits. In practice, it is more convenient to use short-circuited stubs. Short-circuited stubs are usually used in preference to open-circuited stubs because an infinite terminating impedance is more difficult to realise than a zero terminating impedance. Radiation from the open end of a stub makes it appear longer than it is, and compensation for these effects makes the use of open-circuited stubs more cumbersome. A short-circuited stub of an adjustable length is much easier to construct than an open-circuited stub.

It is also more common to connect these stubs in parallel with the main line. For parallel connections, it is convenient to use admittances rather than impedances. In thee cases, we use the Smith chart as an admittance chart to design the matching networks.

A single-stub matching circuit is depicted in Figure 12. Note that the short-circuited stub is connected in parallel with the main line. In order to match the complex load impedance $Z_L$ to the characteristic impedance of the lossless main line, $Z_0 = R_0$, we need to determine the lengths *d* and *l*.

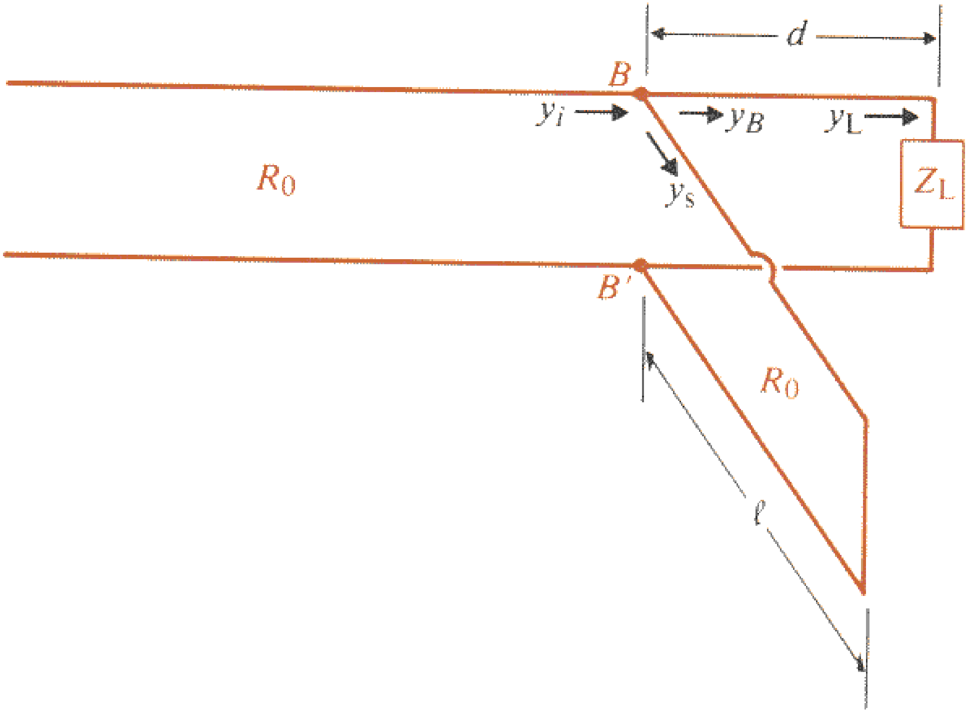



Figure 12: Impedance matching by single stub method.

For the transmission line to be matched at the point $B-B'$, the basic requirement is

$$Y_i = Y_B + Y_s \\ = Y_0 = \frac{1}{R_0}. \quad (20)$$

In terms of normalised admittances, (23) becomes

$$y_i = y_B + y_s = 1. \quad (21)$$

where $y_B = g_B + jb_B = Y_B / Y_0$ for the load section and $y_s = Y_s / Y_0$ for the short-circuited stub. Note that $y_s = -j\cot(2\pi l/\lambda)$ is purely imaginary. It can therefore only contribute to the imaginary part of $y_i$. The position of $B-B'$ (or, in other words, the length *d*) must be chosen such that $g_B = 1$, i.e.

$$y_B = 1 + jb_B. \quad (22)$$

Next, the length *l* is chosen such that

$$y_s = -jb_B, \quad (23)$$

which yields $y_i = y_B + y_s = (1 + jb_B) + (-jb_B) = 1$. The circuit is therefore matched at $B-B'$, and at any point left of $B-B'$ as well.

If we use the Smith chart, we would rotate on a $|\Gamma|$-circle in a clockwise direction (towards the generator) when transforming the normalised load admittance to the admittance $y_B$. However, according to (23), $y_B$ must also be located on the $g = 1$ circle.

The use of the Smith chart for the purpose of designing a single-stub matching network is best illustrated by means of an example.

**Example 5**: A 50 Ω transmission line is connected to a load impedance $Z_L = 35 - j37.5\,\Omega$. Find the position and length of a short-circuited stub required to match the load at a frequency of 200 MHz. Assume that the transmission line is a co-axial line with a dielectric for which $\varepsilon_r = 9$.

Given $Z_0 = R_0 = 50\,\Omega$ and $Z_L = 35 - j47.5\,\Omega$. Therefore $z_L = Z_L / Z_0 = 0.7 - j0.95$ .

- Enter the Smith chart at $z_L$ shown as point $P_1$ in Figure 13.
- Draw a $|\Gamma|$-circle centred at $O$ with radius $\overline{OP_1}$ .
- Draw a straight line from $P_1$ through $O$ to point $P_2'$ on the perimeter, intersecting the $|\Gamma|$-circle at $P_2$ , which represents $y_L$ . Note 0.109 at $P_2'$ on the "wavelengths toward generator" scale.
- Note the two points of intersection of the $|\Gamma|$-circle with the $g = 1$ circle:
  - At $P_3$: $y_{B1} = 1 + j1.2 = 1 + jb_{B1}$
  - At $P_4$: $y_{B2} = 1 - j1.2 = 1 + jb_{B2}$
- Solutions for the position of the stub:
  - For $P_3$ (from $P_2'$ to $P_3'$) $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$
  - For $P_4$ (from $P_2'$ to $P_4'$) $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$
- Solutions for the length of the short-circuited stub to provide $y_s = -jb_B$ :
  - For $P_3$ (from $P_{sc}$ on the extreme right of the admittance chart to $P_3''$ , which represents $y_s = -jb_{B1} = -j1.2$ ): $l_1 = (0.361 - 0.250)\lambda = 0.111\lambda$
  - For $P_4$ (from $P_{sc}$ on the extreme right of the admittance chart to $P_4''$ , which represents $y_s = -jb_{B2} = j1.2$ ): $l_2 = (0.139 + 0.250)\lambda = 0.389\lambda$

To compute the physical lengths of the transmission line sections, we need to calculate the wavelength on the transmission line. Therefore

$$\lambda = \frac{u_p}{f} = \frac{1/\sqrt{\mu\varepsilon}}{f} = \frac{c/\sqrt{\varepsilon_r}}{f} \approx 0.5\ \text{m}.$$

Thus:

| | |
|---|---|
| $d_1 = 0.059\lambda = 29.5$ mm | $l_1 = 0.111\lambda = 55.5$ mm |
| $d_2 = 0.223\lambda = 111.5$ mm | $l_2 = 0.389\lambda = 194.5$ mm |

Note that either of these two sets of solutions would match the load. In fact, there is a whole range of possible solutions. For example, when calculating $d_1$, instead of going straight from $P_2'$ to $P_3'$, we could have started at $P_2'$, rotated clockwise around the Smith chart $n$ times (representing an additional length of $n\lambda/2$ ) and continued on to $P_3'$ , yielding $d_1 = 0.059\lambda + n\lambda/2$, $n = 0, 1, 2, \ldots$ The same argument applies for $d_2$, $l_1$ and $l_2$.

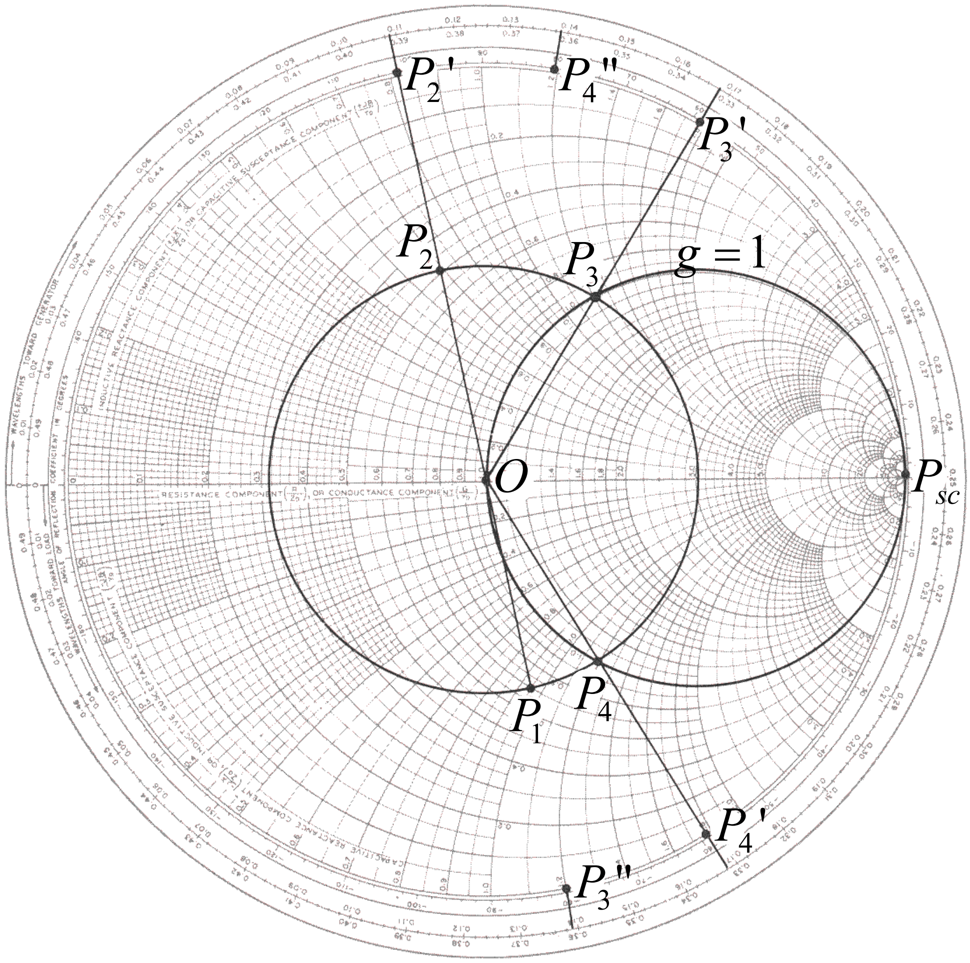



Figure 13: Single-stub matching on an admittance chart (Example 5).