08/02/24

UNIT-I

Proposition :- Propositions (or) a statement is a declarative sentance which is either "TRUE" (or) "FALSE" but not both.

→ The truth or falsity of a proposition is called its truth-value.

→ these two values 'true' & 'false' are denoted by the symbols T & F respectively.

→ It can also denoted as.

$\boxed{\text{true}-1}$, $\boxed{\text{false}-0}$

Example:

① Delhi is the capital of India

② Kolkata is a country.

③ 2+3=4.

} Statements

the above examples are ~~true~~ of ~~false~~ prepositions (or) statements. either it is true (or) false.

Example : (2)

(1) How beautiful are you?
(2) $x+y=8$
(3) Take one book.

$\left.\right\}$ not Statement.

these are not prepositions as they are not declarative in nature, i.e., they do not declare a definite truth value T or F.

⇒ Prepositional Calcus:-

It is also known as Statement calcus. It is a branch of mathematics that is used to describe a logical system or structure. A logical system consists of. (1) A Universe of prepositions
(2) Truth tables (Axioms) for the logical operator
(3) Definition that explain equivalence & implication of preposition

→ LOGIC CONNECTIVITIES:-

We can construct complicated statements from simple statements by using certain connecting words. (or) expressions like.

"AND", "NOR", "OR", "NOT", "NAND", "IF, THEN"

"IF AND ONLY IF", such words. are called connectivities.

→ the statement that we consider initially is simple statement / Atomic statement / Primary / Primitive Statement.

→ the statements obtained by the use of connectivities is called "Molecular statement" (or) Compound statement.

	Name	Ref	Meaning.
1)	Negation	NP/~p	not p
2)	Conjunction	$p \wedge q$	p and q
3)	Disjunction	$p \vee q$	p or q
4)	Exclusive/implication	$p \to q$, $p \oplus q$	either p, q, not both
5)	Biconditional	$p \leftrightarrow q$	p if and only if q

CONNECTIVES :

The word | phrases | symbols which are used to make a proposition by two or more propositions are called logical connectives or simply connectives.

→ There are five basic connectives they are.
① Negation
② Conjuction
③ Disjunction
④ Conditional &
⑤ Bi-conditional.

1) NEGATION :-

A statement obtained by inserting the word 'NOT' at an appropriate place in the given statement.

→ If p denotes a statement then the negation of p is written as ~p & read as _not p' ; ~p.

→ If the truth value of p is T then the truth value of ~p is F. Also if the truth value of p is F then the truth value of ~p is T.

Truth table for negation

p	~p
T	F
F	T

→ Negation is denoted as '~'.

2) CONJUNCTION :-

A compound statement is obtained by combining 2 given statements by inserting a word 'AND' in b/w them is called conjunction of the given statement.

Rule

→ The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T. Otherwise it has truth value F

Truth table for Conjunction

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

→ Conjunction is denoted as '∧'

3) DISJUNCTION:-

this statement is obtained by inserting a word OR in between two statements.

Rule:-

→ the statement p∨q has the truth value F only when both p and q have the truth value F. Otherwise it has truth value T.

TRUTH TABLE FOR DISJUNCTION.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

→ Disjunction is denoted as 'V', read as ~~p∨q~~ "P OR Q".

4) CONDITIONAL :-

If p and q are any two statements then the statement p→q which is read as "If p then q" is called a conditional statement (or) Implication and the connective is the conditional connective.

Rule: the statement p→q has the truth value F only when p is T and q is F & rest of the cases T.

TRUTH TABLE FOR CONDITIONAL

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

→ Conditional is denoted as '→' read as p implies q.

5) BI-CONDITIONAL :-

If p and q are any two statements, then the statement p⟺q which is read as p if & only if q and abbrivated as p iff q is called as biconditional statement & the connective is the biconditional connect

Rule: the statement $p \Leftrightarrow q$ is true when p & q are same that is both are false or both are true.

Truth table for Bi-conditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

→ Bi-conditional is denoted as '$\leftrightarrow$'.
read as p if and only if q.

Rule: the statement $p \Longleftrightarrow q$ is true when p & q are same that is both are false or both are true.

Truth table for Bi-conditional

p	q	$p \Longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

→ Bi-conditional is denoted as '$\Longleftrightarrow$'. read as p if and only if q.

EXAMPLES FOR CONNECTIVES

1) Negation:-

i) p: Kolkata is a city.
   $\sim p$: Kolkata is not a city.

ii) p: 3 is prime number
   $\sim p$: 3 is not a prime number.

2) Conjunction:-

i) p: It is raining today.
   q: There are 10 chairs in the room.
   $p \wedge q$: It is raining today and there are 10 chairs in the room.

ii) p: $\sqrt{2}$ is irrational number.
   q: 9 is multiple of 3.
   $p \wedge q$: $\sqrt{2}$ is irrational number and 9 is multiple of 3.

⇒ Translate the following statement:

① 'Jack and Jill went up the hill' into symbolic form using conjunction

Sol. Let p: Jack went up the hill.
   q: Jill went up the hill. $\rightarrow p \wedge q$
   $p \wedge q$: Jack and Jill went up the hill.

3) Disjunction:-

~~i) p: I shall go the~~

i) p: I shall go to the game.
   q: I shall watch the game on television.
   $p \vee q$: I shall go to the game or watch the game on television.

ii) p: All triangles are equilateral.
   q: 2+5=7.
   $p \vee q$: All triangles are equilateral or 2+5=7.

4) Conditional:-

i) p: 2 is prime number.
   q: 3 is prime number.
   $p \rightarrow q$: if 2 is prime number then 3 is prime number.

→ Translate the following statement into symbolic form using conditional connective.

① 'The crop will be destroyed if there is a flood'

Sol: Let p: the crop will be destroyed.

q: there is a flood.

$p \leftrightarrow q$.

$\Rightarrow$ Let us rewrite the given statement as If there is a flood, then the crop will be destroyed. So, the symbolic form of the given statement is $q \to p$.

(2) Let p & q denote the statements
p: You drive over 70 km per hour.
q: You get a speeding ticket.
Write the following statements into symbolic form.

i) You will get a speeding ticket if you drive over 70 km per hour.

Sol: $p \to q$.

ii) Whenever you get a speeding ticket, you drive over 70 km per hour.

Sol: $q \to p$.

(ii) p: It rains
q: the crop will grow
$p \to q$: If it rains then the crop will grow

6) NAND (↑):

NAND operation is combination of both NOT + AND.

→ It is denoted by ↑

→ $p \uparrow q$ is false when two statements are true.

truth table for NAND

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

7) NOR (↓):-

NOR is combination of NOT + OR

→ It is denoted by ↓

→ $p \downarrow q$ is true when two statement are false. the truth table follows.

truth table for NOR

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

8) Exclusive OR (⊕):-

Let p & q are preposition, the exclusive OR of p & q is denoted by $p \oplus q$. is the preposition that is true when exactly one of p & q is true.

truth table for Exclusive OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

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Example problems

1) Prove that $p \wedge (p \vee q)$

Sol:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

2) Find the truth table for

i) $p \wedge (\sim p \vee q)$

ii) $\sim p \wedge (\sim p \vee \sim q)$

Sol: i)

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

ii) $\sim p \wedge (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim p \wedge (\sim p \vee \sim q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	T

3) $p \wedge (p \to q)$

p	q	$p \to q$	$p \wedge (p \to q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

4) $(p \wedge q) \Leftrightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \Leftrightarrow p \vee q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	F	T

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Some more example problems.

1) $(p \vee q) \vee \sim p$
2) $(p \wedge (p \rightarrow q)) \rightarrow q$
3) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
4) $(q \wedge (p \rightarrow q) \rightarrow p$
5) $\sim (p \rightarrow q) \rightarrow \sim p$
6) $(p \vee q) \wedge (p \rightarrow q)$
7) $((p \rightarrow q) \wedge (q \rightarrow R)) \rightarrow (p \rightarrow R)$
8) $(p \wedge (\sim q))$
9) $p \vee (\sim q)$
10) $p \rightarrow (q \vee R)$

→ Propositional Equivalence

Propositional equivalence is of 4 types they are:

1) Tautology.
2) Contradiction.
3) Contingency.
4) logical Equivalence.

1) Tautology :-

A compound proposition which is true for all possible truth values of its components is called tautology. Denoted by T_0.

Eg:- P.T $[(p \rightarrow q) \wedge p] \rightarrow q$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

As we can see every value of $((p \rightarrow q) \wedge p) \rightarrow q$ is true. It is tautology.

2) Contradiction:

A Compound preposition, which is 'false' for all possible truth values of its components is called Contradiction

→ Denoted by F_0

Ex - P.T $(p \vee q) \wedge [(\sim p) \wedge (\sim q)]$

p	q	p∨q	~p	~q	~p∧~q	(p∨q)∧[~p∧~q]
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

As we can say every value of $(p \vee q) \wedge [\sim p \wedge \sim q]$ is false. It is Contradiction.

3) Contingency:

A statement formula which is neither a tautology nor a Contradiction is known as a Contingency.

Ex :- Prove $(p \vee q) \wedge (\sim p)$

p	q	p∨q	~p	(p∨q)∧~p
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

4) logical Equivalence:-

two statements / propositions are said to be logically equivalent whenever P, Q have same truth values.

→ Represent as "$P \equiv Q$", "EPQ", "$P \Leftrightarrow Q$"

→ Let 2 statements x & y are logically equivalent if any of the following 2 condition satisfy.

1) The truth table of each statement have the same truth values.

2) the Biconditional statement x if & only if y is a tautology $(x \Leftrightarrow y)$

Eg:- $(p \to q) \Leftrightarrow (\sim p) \vee q$.

p	q	$p \to q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As we can see it has same truth values.

→ $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

2) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

3) $(p \to q) \wedge (\sim p \to \sim q)$

4) $(p \vee q) \to (p \wedge q)$

→ Equivalence Formulas:

Laws of Logics:-

1) Negation :-

Law of double negation for any statement P

$\sim(\sim p) \Leftrightarrow p$.

2) Idempotent law: for any proposition p.

$(p \vee p) \Leftrightarrow p$

$(p \wedge p) \Leftrightarrow p$

3) Identity law:-

$(p \vee F_0) \Leftrightarrow p$

$(p \wedge T_0) \Leftrightarrow p$

4) Inverse law:-

$(p \vee \sim p) \Leftrightarrow T_0$

$(p \wedge \sim p) \Leftrightarrow F_0$

5) Domination law:-

$(p \vee T_0) \Leftrightarrow T_0$

$(p \wedge F_0) \Leftrightarrow F_0$

6) Commutative Laws:- for statements p&q

$(p \vee q) \Leftrightarrow (q \vee q)$

$(p \wedge q) \Leftrightarrow (q \wedge p)$

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7) **Absorption law:-**

$(p \vee (p \wedge q)) \Leftrightarrow p$

$(p \wedge (p \vee q)) \Leftrightarrow p$

8) **Demorgan's law:-**

$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

9) **Associative law:-** for three statements p, q, r

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

10) **Distributive law:-**

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

11) **Conditional:-**

$\sim(p \to q) \Leftrightarrow p \wedge \sim q$

$(p \to q) \Leftrightarrow \sim p \vee q$

* Table for negation of compound Statement

Statement	Negation
$\sim p$	p
$p \wedge q$	$\sim p \vee \sim q$
$p \vee q$	$\sim p \wedge \sim q$
$p \to q$	$p \wedge \sim q$

* logical equivalence involving conditional statement. (Implication (or) If-then)

p	q	$p \to q$	$\sim p \vee q$
T	T	T	T
T	F	F	F

1) $p \to q \equiv (\sim p \vee q)$
2) $p \to q \equiv \sim q \to \sim p$
3) $p \vee q \equiv \sim p \to q$
4) $p \wedge q \equiv \sim(q \to \sim p)$
5) $\sim(p \to q) \equiv p \wedge \sim q$
6) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.

* logical equivalence Involving Bi-Conditional Statement

p	q	~p	~q
T	T	F	F
T	F	F	T

1) $p \Leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

2) $p \Leftrightarrow q \equiv \sim p \Leftrightarrow \sim q$

3) $p \Leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

4) $\sim (p \Leftrightarrow q) \equiv p \Leftrightarrow \sim q$

Example problems:

1) "If x is not a real number, then it is not a rational number & not an irrational number."

Sol: let p : x is real number.
q : x is rational number.
r : x is irrational number.

$[\sim p \to (\sim q \wedge \sim r)]$

$\sim [\sim p \to (\sim q \wedge \sim r)] \equiv \sim p \wedge [\sim (\sim q \wedge \sim r)]$

$\equiv \sim p \wedge (\sim\sim q \vee \sim\sim r)$

$\equiv \sim p \wedge (q \vee r)$

∴ negation of given preposition is

"x is not a real number and it is a rational number & an irrational number".

Example:

1) S.T $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent:

p	q	~p	~q	p∨q	~(p∨q)	~p∧~q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

therefore $\sim(p \vee q) \equiv \sim p \wedge \sim q$

2) Prove that $(p \vee q) \wedge [\sim ((\sim p) \wedge q)]$ is logically equivalent using law of logic.

$(p \vee q) \wedge [\sim ((\sim p) \wedge q)]$

$(p \vee q) \wedge [(\sim (\sim p)) \vee \sim q]$ demorgan law

$(p \vee q) \wedge [p \vee \sim q]$ Double negation law

$p \vee (q \wedge \sim q)$ distributive law

$p \vee F$ Inverse law

p Identity law.

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5) Prove that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$ ...

Sol: $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

$\Rightarrow ((\neg p \wedge \neg q) \wedge r) \vee ((q \vee p) \wedge r)$

(Associative & distributive law)

$\Rightarrow (\neg(p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$

[de morgan's law]

$\Rightarrow (\neg(p \vee q) \vee (p \vee q)) \wedge r$

[distributive law]

$\Rightarrow T \wedge R$ [$\because \neg P \vee P \Leftrightarrow T$]

$\rightarrow R$ //

4) Show $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is tautology

Sol: By De Morgan's law, we have

$\neg p \wedge \neg q \Leftrightarrow \neg(p \vee q)$

$\neg p \vee \neg r \Leftrightarrow \neg(p \wedge r)$

therefore,

$(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r) \Leftrightarrow \neg(p \vee q) \vee \neg(p \vee r)$

$\Rightarrow \neg(p \vee q) \vee \neg(p \vee r)$

$\Rightarrow \neg((p \vee q) \wedge (p \vee r))$

also

$\neg(\neg p \wedge (\neg q \vee \neg r))$

$\Rightarrow \neg(\neg p \wedge \neg(q \wedge r))$

$\Rightarrow p \vee (q \wedge r)$

$\Rightarrow (p \vee q) \wedge (p \vee r)$

Hence

$((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r)))$

$\Rightarrow (p \vee q) \wedge (p \vee q) \wedge (p \vee r)$

$\Rightarrow (p \vee q) \wedge (p \vee r)$

thus

$((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$\Rightarrow [(p \vee q) \wedge (p \vee r)] \vee \neg[(p \vee q) \wedge (p \vee r)]$

$\Rightarrow T$

Hence the given formula is a tautology.

5) Show that $(p\wedge q)\rightarrow(p\vee q)$ is a tautology.

Sol: $(p\wedge q)\rightarrow(p\vee q)$

$\Rightarrow \neg(p\wedge q)\vee(p\vee q)$. $[\because p\rightarrow q \Leftrightarrow \neg p\vee q]$

$\Rightarrow (\neg p\vee\neg q)\vee(p\vee q)$ [by De Morgan's law]

$\Rightarrow (\neg p\vee p)\vee(\neg q\vee q)$ [by Associative law & Commutative law]

$\Rightarrow (T\vee T)$ [by negation laws]

$\Rightarrow T$

Hence, the result.

6) Prove that $\sim[\sim((p\vee q)\wedge r)\vee\sim q]$

Sol: $\sim(\sim((p\vee q)\wedge r)\vee\sim q)$

$\Rightarrow \sim\{\sim((p\vee q)\wedge r)\wedge\sim(\sim q)\}$

$\Rightarrow \sim(\sim((p\vee q)\wedge r)\wedge q)$

($\because$ de morgans law)

$\Rightarrow ((p\vee q)\wedge r)\wedge q$

($\because$ double negation law)

$\Rightarrow (p\vee q)\wedge(q\wedge r)$ (using associative, Commutative law)

$\Rightarrow ((p\vee q)\wedge q)\wedge r$, (associative law)

$\Rightarrow q\wedge r$. (using absorption law).

7) $[p\vee(p\wedge(p\vee q))]\Leftrightarrow p$

Sol: $p\vee p$ (Absorption law)

p (Idempotent law).

8) $(p\vee F)\wedge(q\vee T)\Leftrightarrow p$

Sol: $p\wedge(q\vee T)$ (Identity law)

$p\wedge T$ (domination law)

$p\equiv p$ (Identity law)

9) $\sim(\sim p)\vee[(p\vee F)\wedge\sim(\sim q)]\Leftrightarrow p$

Sol: $p\vee[(p\vee F)\wedge q]$ [double negation law]

$p\vee[p\wedge q]$ [Identity law]

p [Absorption law].

10) $[\sim(p\wedge q)\wedge q]\Leftrightarrow(\sim p\vee q)$

11) P.T $(p\vee q)\wedge\sim(\sim p)\vee q)$

12) $\sim(\sim((p\vee q)\wedge r)\vee\sim q)$

13) $((p\rightarrow q)\rightarrow r)\Leftrightarrow(\sim p\vee q)\rightarrow r$.

14) $p\vee(p\wedge(p\vee q))\Leftrightarrow p$.

15) $[(p\to q)\wedge(\sim q\wedge(q\vee\sim q))] \Leftrightarrow \sim(q\vee p)$

Sol: $[(p\to q)\wedge(\sim q\wedge(\sim q\vee q))]$ (commutative law)

$\Rightarrow (p\to q)\wedge\sim q$ [$\because$ Absorption law]

$\Rightarrow (\sim[(p\to q)\to q]$ [$\because \sim(p\to q)\Leftrightarrow(p\wedge\sim q)$]

$\Rightarrow \sim(\sim(p\to q)\vee q)$ [$\because (p\to q)\Leftrightarrow(\sim p\vee q)$]

$\Rightarrow \sim[(p\wedge\sim q)\vee q]$

$\Rightarrow \sim[q\vee(p\wedge\sim q)]$ [$\because$ Commutative law]

$\Rightarrow \sim[(q\vee p)\wedge(q\vee\sim q)]$ [Distributive law]

$\Rightarrow \sim((q\vee p)\wedge T_0)$. [$q\wedge\sim q$ is tautology].

$\Rightarrow \sim(q\vee p)$ [identity law].

Converse, Inverse & Contrapositive.

→ **Inverse**

Implication of if then is also called a conditional statement i.e. $p\to q$. It has two parts.

$\boxed{p\to q \text{ is } \sim p\to\sim q}$

i) Hypothesis (p)

ii) Conclusion (q)

→ An Inverse of the conditional statement is the negation of both the hypothesis & the conclusion.

→ If the statement is "$\underset{p\to q}{\underline{\text{If } p, \text{ then } q}}$", the Inverse will be "If not p, then not q" ($\sim p\to\sim q$).

① Sol: p: If you do your home work

q: you will not be punished.

Inverse: p: If you do not do your home work.

q: then you will be punished.

② Sol: p: Today is Sunday

q: I will go for a walk.

Inverse: p: If today is not Sunday.

q: then I will not go for a walk.

2) Converse:

The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. $(p \to q)$

→ If the statement is "If p, then q" then converse will be "If q, then p" $(q \to p)$

→ the converse of $p \to q$ is $q \to p$.

① Ex: p: If you do your homework, then q: you will not be punished

Converse: If You will not ~~be~~ get punished, then you will ~~no~~ do your homework.

② Ex: Today is Sunday. I will go for a walk.

Converse: If I ~~will~~ am going for a walk then it is Sunday.

3) Contrapositive

The contrapositive of the conditional statement is computed by interchanging the hypothesis and conclusion of Inverse statement

→ If the statement is "If p, then q" the contrapositive "If not q, then not p"

→ the contrapositive of $\boxed{p \to q \text{ is } \sim q \to \sim p}$

Ex: If you do your homework. You will not be punished.

Contrapositive: If you are punished, then you did not do your homework.

② Ex: Today is Sunday. I will go for a walk

Contrapositive: If I will not go for a walk then today is not Sunday.

Converse, Inverse, Contrapositive.

consider a $p \to q$, then

1) $q \to p$ is called converse of $p \to q$.
2) $\sim p \to \sim q$ is called inverse of $p \to q$.
3) $\sim q \to \sim p$ is called contrapositive of $p \to q$.

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$\therefore p \to q \Leftrightarrow (\sim q \to \sim p)$.

$q \to p \Leftrightarrow (\sim p \to \sim q)$.

Exercise

1) State the converse, inverse & contrapositive of the following conditionals.

i) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Sol: <u>Converse</u>:- If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

<u>Inverse</u>: If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

<u>Contrapositive</u>: If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

ii) If a real number x^2 is greater than zero, then x is not equal to zero.

Sol: <u>Converse</u>: If a real number x is not equal to zero, then x^2 is greater than zero.

<u>Inverse</u>: If a real number x^2 is not greater than zero, then x is equal to zero.

Contrapositive: If a real number x is equal to zero, then x^2 is not greater than zero.

iii) If a triangle is not isosceles, then it is not equilateral.

Converse: If a triangle is not equilateral, then it is not isosceles.

Inverse: If a triangle is isosceles, then it is equilateral.

Contrapositive: If a triangle is equilateral then it is isosceles.

2) Write down the conditionals given in the previous exercise by using

a) the 'necessary condition' language

b) the 'sufficient condition' language

Sol:

1) A necessary condition for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

2) A necessary condition for real number x^2 to be greater than zero is that x is not equal to zero.

3) For a triangle to be non-isosceles it is necessary that it is not equilateral.

→ Normal form :-

Conjunction → product → AND → $\wedge$

Disjunction → Sum → OR → $\vee$

→ It means of representing propositional expression.

→ Here, we use the words (product, Sum)

↳ Elementary product :-

We call it as fundamental conjunction.

→ A product of variables & their negation in a formula is called Elementary product.

$P \wedge Q$

$\sim P \wedge Q$

$P \wedge \sim Q$

$\sim P \wedge \sim Q$

↳ Elementary Sum :- Fundamental disjunction

→ Sum of variables & their negation is called Elementary Sum.

$P \vee Q$ $\quad$ $P \vee \sim Q$

$\sim P \vee Q$ $\quad$ $\sim P \vee \sim Q$

→ Their are two types of normal form. they are :-

① Disjunction Normal form (DNF)

② Conjunction Normal form (CNF)

1) DNF :-

A formula which is equivalent to given formula & which consists of "Sum of elementary products".

$\boxed{\underbrace{(P \wedge Q)}_{\text{Prod}} \underbrace{\vee}_{\text{Sum}} (\sim P \underbrace{\wedge}_{\text{Product}} \sim Q)}$

2) CNF :-

A formula which is equivalent to given formula & which consists of "Product of Sum terms" is called CNF

$\boxed{(P \vee Q) \wedge (\sim P \vee \sim Q)}$

Procedure :-

1) Replace implication, Bi-implication by an equivalent expression containing $\wedge, \vee, \neg$ only.

Ex:- $p \to q \equiv (\sim p \vee q)$

$p \Leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

2) Eleminate negation befor sum & product by using double negation or Demorgan's law.

3) Apply distributive law until DNF or CNF is obtained.

* Example problem for DNF :-

1) Find DNF of $P \wedge (P \to Q)$.

Sol: Given,

$P \wedge (P \to Q)$.

$P \wedge (\sim P \vee Q)$ (∵ conditional law)

$(P \wedge \sim P) \vee (P \wedge Q)$ (∵ distributive law)

∴ It is in DNF form.

$\boxed{(P \wedge Q) \vee (\sim P \wedge \sim Q)}$

2) $\sim [P \to (Q \wedge R)]$

Sol: Given,

$\sim [P \to (Q \wedge R)]$. $\boxed{P \to q \equiv \sim p \vee q}$

$\Rightarrow \sim [(\sim P) \vee (Q \wedge R)]$ [∵ conditional law]

$\Rightarrow P \wedge \sim (Q \wedge R)$ [∵ double negation law].

$\Rightarrow P \wedge (\sim Q \vee \sim R)$. [∵ demorgan's law]

$\Rightarrow (P \wedge \sim Q) \vee (P \wedge \sim R)$. [∵ distributive law]

∴ It is in DNF form.

* Example problem for CNF :-

1) Find out CNF of $\sim [P \wedge (P \to Q)]$.

Sol: Given,

$\sim [P \wedge (P \to Q)]$.

$\Rightarrow \sim P \vee \sim (P \to Q)$ [∵ Demorgan's law].

$\Rightarrow \sim P \vee \sim ((\sim P) \vee Q)$ [∵ conditional law].

$\Rightarrow \sim P \vee (P \wedge \sim Q)$ [∵ Demorgan's law]

$\Rightarrow (\sim P \vee P) \wedge (\sim P \vee \sim Q)$ [∵ Distributive law].

∴ It is in CNF form.

$\boxed{(P \vee Q) \wedge (\sim P \vee Q)}$

2) $\sim((\sim P \to \sim Q) \wedge R)$

Sol) $\sim[\sim P \to \sim Q] \vee \sim(\sim R)$ [Demorgan's laws]

$\Rightarrow \sim[\sim(\sim P) \vee \sim Q] \vee R$ [∵ Conditional laws]

$\Rightarrow \sim(\sim P \wedge Q) \vee R$ — [∵

$\Rightarrow \sim(\sim P \to \sim Q) \vee R$

$\Rightarrow (\sim P \wedge \sim(\sim Q)) \vee R$

$\Rightarrow (\sim P \vee R) \wedge (Q \vee R)$ (∵ Distributive laws)

∴ It is in CNF form.//

→ Principle Normal forms

Given two simple prepositions p & q. the compound prepositions are of two types ① Minterms & ② Max terms.

def → backside

* Minterms:

For 2 prepositions (p, q) ($2^2 = 4$).

$p \wedge q$

$\sim p \wedge q$

$p \wedge \sim q$

$\sim p \wedge \sim q$ min

for 3 prepositions (p, q, r) ($2^3 = 8$)

$p \wedge q \wedge r$ $\sim p \wedge \sim q \wedge \sim r$

$\sim p \wedge q \wedge r$ $p \wedge \sim q \wedge r$

$\sim p \wedge \sim q \wedge r$ $p \wedge q \wedge \sim r$

$\sim p \wedge \sim q \wedge \sim r$ $\sim p \wedge q \wedge \sim r$

→ these are the terms involving p & q.

* Maxterms:-

the duals of these minterms.

for 2 preposition (p, q)

$p \vee q$

$\sim p \vee q$

$p \vee \sim q$

$\sim p \vee \sim q$

for 3 prepositions (p, q, r).

$p \vee q \vee r$ $p \vee \sim q \vee \sim r$

$\sim p \vee q \vee r$ $p \vee \sim q \vee r$

$\sim p \vee \sim q \vee r$ $p \vee q \vee \sim r$

$\sim p \vee \sim q \vee \sim r$ $\sim p \vee q \vee \sim r$

(3) Principle normal form consists of 2 types. they are.
(1) Principle Disjunctive normal form (PDNF)
(2) Principal Conjunctive normal form (PCNF)

(4) the DNF & CNF of statement formula is not unique in order to obtain a unique result of a given statement formula we introduce principle normal form.

1) Principal Disjunctive Normal form: (PDNF):

A compound proposition is involving two simple propositions p and q. an equivalent compound proposition 'v' consisting of disjunction of the minterms involving p and q only is known as its principal disjunctive normal form. (or sum-of-products canonical form).

Min terms:

For a given no of variables, the minterm consists of conjunctions in which each statement variable or its negation but not both, appears only once.

P	Q	$P \wedge Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

→ From the above truth tables of these minterms of P & Q, it is clear that.
i) no two minterms are equivalent.
ii) Each minterm has the truth value T for ~~every~~ exactly one combination of the truth values of the variables P & Q

* Obtain PDNF of a given formula by truth table

i) Construct a truth table of the given formula

ii) for every truth value T ~~has the value~~ in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of ~~good~~ P and Q.

iii) the disjunction of these minterms will then be equivalent to the given formula.

~~⇒ The PDNF of $P \to Q$ is $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$.~~

~~⇒ $P \to Q = (P \wedge Q) \wedge (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$~~

Example. Obtain the PDNF of $P \to Q$.

Solution: from the truth table of $P \to Q$.

P	Q	P→Q	Minterm
T	T	T	$P \wedge Q$ ✓
T	F	F	$P \wedge \sim Q$
F	T	T	$\sim P \wedge Q$ ✓
F	F	T	$\sim P \wedge \sim Q$ ✓

→ The PDNF of $P \to Q$ is $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

$\therefore P \to Q \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

Example: Obtain the PDNF for $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R) \Rightarrow S$

Sol:

P	Q	R	Minterm	P∧Q	~P∧R	Q∧R	S
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \sim R$	T	F	F	T
T	F	T	$P \wedge \sim Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \sim Q \wedge \sim R$	F	F	F	F
F	T	T	$\sim P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\sim P \wedge Q \wedge \sim R$	F	F	F	F
F	F	T	$\sim P \wedge \sim Q \wedge R$	F	T	F	T
F	F	F	$\sim P \wedge \sim Q \wedge \sim R$	F	F	F	F

∴ the PDNF of $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$ is

$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \dots$

(b) Without constructing the truth table:

In order to obtain the PDNF of given formula is constructed as follows.

1) First replace $\rightarrow$, by their equivalent formula containing only $\wedge, \vee$ and $\sim$

2) Next negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.

3) Any elementarily product which is a contradiction is dropped.

→ Minterms are obtained in the disjunction by introducing the missing factors.

→ Identical minterms appearing in the disjunctions are deleted.

Example: Obtain the PDNF of $P \rightarrow Q$

(a) $\sim P \vee Q$

(b) $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

(a) $P \rightarrow Q \Rightarrow$

$\sim P \vee Q \Leftrightarrow (\sim P \wedge T) \vee (Q \wedge T)$ [∵ $P \wedge T \Leftrightarrow P$]

$\Rightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (Q \wedge (P \vee \sim P))$

[∵ $P \vee \sim P \Leftrightarrow T$]

$\Rightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (Q \wedge P) \vee (Q \wedge \sim P)$

[∵ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$]

$\Rightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q)$ [∵ $P \vee P \Leftrightarrow P$]

(b) $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

$\Rightarrow (P \wedge Q \wedge T) \vee (\sim P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$

$\Rightarrow (P \wedge Q \wedge (R \vee \sim R)) \vee (\sim P \wedge R \wedge (Q \vee \sim Q)) \vee (Q \wedge R \wedge (P \vee \sim P))$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R)$

$\Rightarrow P \vee (P \wedge Q) \Leftrightarrow P$

$\Rightarrow P \vee (\sim P \wedge Q) \Leftrightarrow P \vee Q$

2) Principal Conjunctive Normal Form:- (PCNF): Max terms:

The dual of a minterm is called a maxterm. for a given number of variables. the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

→ Each of the maxterm has the truth value F for exactly one combination of the truth values of the variables.

Q) Principal Conjunctive Normal Form (PCNF):-

A Compound proposition i involving two simple propositions p and q, an equivalent compound proposition i consisting of conjunctions of the maxterms involving p and q only is known as its

Principal Conjunctive normal form (or product - of - sums Canonical form)

→ the method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

→ Example: Obtain the PCNF of the formula $(\sim P \to R) \wedge (Q \leftrightarrow P)$

Sol: $(\sim P \to R) \wedge (Q \leftrightarrow P)$

$\Rightarrow (\sim(\sim P) \vee R) \wedge ((Q \to P) \wedge (P \to Q))$

$\Rightarrow (P \vee R) \wedge [(\sim Q \vee P) \wedge (\sim P \vee Q)]$

$\Rightarrow (P \vee R \vee F) \wedge [(\sim Q \vee P \vee F) \wedge (\sim P \vee Q \vee F)]$

$\Rightarrow [(P \vee R) \vee (Q \wedge \sim Q)] \wedge [(\sim Q \vee P) \vee (R \wedge \sim R)]$
$\wedge [(\sim P \vee Q) \vee (R \wedge \sim R)]$

$\Rightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \sim Q) \wedge (P \vee \sim Q \vee R)$
$\wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q$

$\Rightarrow (P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee R)$
$\wedge (\sim P \vee Q \vee \sim R)$

$\therefore$ It is required PCNF.

Example: Obtain the prime PCNF of the following.

i) $(\sim p \to q) \wedge (q \leftrightarrow p)$.

ii) $(p \wedge q) \vee (\sim p \wedge q)$

Sol: i) $(\sim p \to q) \wedge (q \leftrightarrow p)$

$(\sim p \to q) \wedge (q \leftrightarrow p)$

$\Rightarrow (p \vee q) \wedge \{(q \to p) \wedge (p \to q)\}$

$\Rightarrow (p \vee q) \wedge \{(\sim q \vee p) \wedge (\sim p \vee q)\}$.

$\Rightarrow (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$.

ii) $(p \wedge q) \vee (\sim p \wedge q)$

$\Rightarrow \{(p \wedge q) \vee \sim p\} \wedge \{(p \wedge q) \vee q\}$

$\Rightarrow (p \vee \sim p) \wedge (q \vee \sim p) \wedge \{(p \vee q) \wedge (q \vee q)\}$

$\Rightarrow T_0 \wedge (q \vee \sim p) \wedge \{(p \vee q) \wedge T_0\}$

$\Rightarrow (q \vee \sim p) \wedge (p \vee q)$

$\Rightarrow$ Obtain PDNF

i) $p \vee q$

Sol: $(p \wedge T_0) \vee (q \wedge T_0)$

$(p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$

$\Rightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$\Rightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$ //

$\Rightarrow$ PCNF using truth tables.

For every truth value 'F' of a given formula, select the maxterms which also has the truth value 'F' for some combination of negation of truth value of the statement formula.

$(p \to q) \wedge (q \to p)$

$(\sim p \vee q) \wedge (\sim q \vee p)$

Sol. $p \Leftrightarrow q$

p	q	$p \Leftrightarrow q$	maxterms
T	T	T	$\sim p \vee \sim q$
T	F	F	$\sim p \vee q$ ✓
F	T	F	$p \vee \sim q$ ✓
F	F	T	$p \vee q$

∴ the equivalent PCNF of $p \Leftrightarrow q$ is $(\sim p \vee q) \wedge (p \vee \sim q)$

→ Without using truth table:

1) Replace the conditional & Bi-conditional connectives in the given statement by ∧, ∨, ~

2) Apply negation to the variable by using demorgan followed by distributive laws.

3) Any elementary sum which is a tautology can be droped. $(p \vee \sim p) = T$.

4) Max terms are obtained in conjunction by introducing missing factors.

5) Identical max terms appearing in the conjunction are deleted.

DNF

1) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$
2) $\sim [p \to (q \wedge r)]$
3) $p \wedge (p \to q)$

CNF

1) $\sim (p \vee q) \Leftrightarrow (p \wedge q)$
2) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$
3) $\sim (p \to q) \vee (r \to p)$

PDNF

1) $\sim p \vee q$
2) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$
3) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$

PCNF

1) $(\sim p \to r) \wedge (q \Leftrightarrow p)$
2) $(\sim p \to q) \wedge (q \Leftrightarrow p)$
3) $(p \wedge q) \vee (\sim p \wedge q)$

→ Well formed formula:

Statement formula:

A statement formula is an expression consisting of connectives, statements & parenthesis is known as well formed formula.

Rules for well formed formula:

1) A statement variable standing alone itself is known as well formed formula.

2) If a statement p is well formed then negation of p is well formed formula

3) If p & q are well formed then $(p \wedge q)$, $(p \vee q)$, $(p \to q)$ & $(p \leftrightarrow q)$ are well formed formula.

Ex:- $\sim(p \vee q)$ it is well formed.

p, q - variables

$\vee$ - connective

( ) - paranthesis

→ Quantifiers:

Quantifiers is nothing but collection of words.

All, for all } $\forall$ there exist, some } $\exists$

→ Theory of Inference for the Statement Calculus.

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as inference theory.

→ If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof & the argument is called a valid argument & conclusion is called a valid conclusion.

Rule of Inference:

Consider a set of propositions $P_1, P_2, P_3 \ldots P_n$, & a proposition C then a compound proposition of the form is $(P_1 \wedge P_2 \wedge P_3 \wedge \ldots P_n) \rightarrow C$ is called an argument.

→ here $P_1, P_2, P_3 \ldots P_n$ are called premises of the argument & C is called conclusion of the argument. therefore the tabular form of the compound proposition is

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \\ \hline C \end{array}$$

Consistency & Inconsistency of premises:

i) Consistency:

A set of premises $H_1, H_2, H_3 \ldots H_m$ are said to be consistent if their conjunction $H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_m$ has the truth value 'T' in atleast one possible situation.

Q. S.T the following premises are consistent.

$(p \vee q) \wedge (\sim p)$.

Sol:

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge (\sim p)$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

In the given statement in result there is atleast one True value.

∴ It is a the above premises are consistent.

2) Inconsistency:

A set of premises H_1, H_2, H_3 are said to be inconsistent if their conjunction $H_1 \wedge H_2 \wedge H_3 \ldots H_n$ has the truth value 'F' (false) in every possible situation is called inconsistency.

Q: S.T the following premises are inconsistent.

$(\sim p \wedge q) \wedge (p)$

p	q	~p	~p∧q	(~p∧q)∧p
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

∴ the above premises are inconsistent because $(\sim p \wedge q) \wedge p$ is false in all possible situations.

→ Valid and Invalid Arguments.

An argument is a set of initial statements are called premises followed by a conclusion.

→ An argument is valid if & only if in every case where all the premises are true otherwise the argument is invalid.

→ An argument with premises $P_1, P_2, P_3, \ldots P_n$ & conclusion c is said to be a valid if whenever each premises $P_1, P_2, P_3 \ldots P_n$ is true then the conclusion c is also true.

→ In other words the argument $(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n) \Rightarrow c$

→ the conclusion is true only in the case of valid argument.

Q1: Is the following argument is valid or not.

Argument - If my computer is crashes then I will loose all my photos.

-> I have not lost all my photos. therefore my computer has not crashed.

next line

Sol: p: my computer crashes

q: I'll loose my photos

: $p \to q$

: $\sim q$

: $\sim p$

Conclusion: $\sim p$

With truth table

$((p \to q) \wedge \sim q) \to \sim p \Rightarrow R$

p	q	~p	~q	p→q	(p→q)∧~q	R
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

∴ the given argument is valid.

Rules of Inference

Rules of Inference	Tautology	Name
1) p, $p \to q$ / q	$[p \wedge (p \to q)] \to q$	Modus ponens (or) law of detachment
2) $\sim q$, $p \to q$ / $\sim p$	$[\sim q \wedge (p \to q)] \to \sim p$	Modus tollens (or) law of Contrapositive
3) $p \to q$, $q \to r$ / $p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism (or) Transitive rule.
4) $p \vee q$, $\sim q$ / q	$[(p \vee q) \wedge \sim q) \to q$	disjunctive syllogism.
5) p / $p \vee q$	$p \to (p \vee q)$ $q \to (p \vee q)$	Addition
6) $p \wedge q$ / p	$(p \wedge q) \to p$ $(p \wedge q) \to q$	Simplification
7) p, q / $p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	conjunction
8) $p \vee q$, $\sim p \vee r$ / $q \vee r$	$[(p \vee q) \wedge (\sim p \vee r)] \to (q \vee r)$	Resolution

→ Theory of Inference for the Statement Calculus

The truth table technique becomes tedious if the premises contains a large no. of statement variables to overcome this limitation we follow other possible methods without using truth table..

There are 3 types of inference rules

1) Rule - P
2) Rule - T
3) Rule - CP

∴ Rule P & T are called the 2 basic rules of Inference.

1) Rule-P:
We may introduce a premises at any step in the derivation.

2) Rule-T:
A formula S may be introduced in the derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

3) Rule-CP:
If a formula S can be derived from another formula R & a set of premises, then the statement $R \to S$ can be derived from the set of premises alone.

* Implication Rules:

I_1: $p \land q \Rightarrow p$ } Simplification
I_2: $p \land q \Rightarrow q$ }

I_3: $p \to p \lor q$ } Addition
I_4: $q \to p \lor q$ }

I_5: $\sim p \Rightarrow p \to q$

I_6: $\sim q \Rightarrow p \to q$

I_7: $\sim (p \to q) \Rightarrow p$

I_8: $\sim (p \to q) \Rightarrow q$

I_9: $p, q \Rightarrow p \land q$

I_{10}: $\sim p, p \lor q \Rightarrow q$ (disjunctive Syllogism)

I_{11}: $p, p \to q \Rightarrow q$ (Modus ponens)

I_{12}: $\sim q, p \to q \to \sim p$ (Modus Tollens)

I_{13}: $p \to q; q \to r \Rightarrow p \to r$ (Hypothetical Syllogism)

I_{14}: $p \lor q, p \to r, q \to r \Rightarrow r$ (dilemma).

* Equivalence Rule

E_1: $\sim\sim p \Rightarrow p$ (Double negation)

E_2: $p \wedge q \Leftrightarrow q \wedge p$ } Commutative.

E_3: $p \vee q \Leftrightarrow q \vee p$

E_4: $(p \wedge q) \wedge r \Leftrightarrow (p) \wedge (q \wedge r)$ } Associative

E_5: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

E_6: $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ } Distributive

E_7: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

E_8: $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$ } DeMorgan's

E_9: $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$

E_{10}: $p \vee p \Leftrightarrow p$.

E_{11}: $p \wedge p \Leftrightarrow p$.

E_{12}: $r \vee (p \wedge \sim p) \Leftrightarrow r$

E_{13}: $r \wedge (p \vee \sim p) \Leftrightarrow r$

E_{14}: $r \vee (p \vee \sim p) \Leftrightarrow T$

E_{15}: $r \wedge (p \wedge \sim p) \Leftrightarrow F$

E_{16}: $p \rightarrow q \Leftrightarrow \sim p \vee q$

E_{17}: $\sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$

E_{18}: $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$

E_{19}: $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

E_{20}: $\sim (p \rightleftarrows q) \Leftrightarrow p \rightleftarrows \sim q$

E_{21}: $p \Leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

E_{22}: $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$.

Example problems.

1) Demonstrate that R is valid inference from the given premises.

$P, p \rightarrow q, q \rightarrow r$.

Sol:

Premises	Step		
{1}	(1)	P	Rule P
{2}	(2)	$p \rightarrow q$	Rule P
{1,2}	(3)	q	Rule T {1,2}
{4}	(4)	$q \rightarrow r$	
{(1,2),4}	(5)	$\therefore r$	Rule T {1,2,4}

$\therefore$ the given premises is valid.

2. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \sim H$, $\sim H \to (A \wedge \sim B)$ and $(A \wedge \sim B) \to R \vee S$

Sol: {1} (1) $C \vee D$ Rule P

{2} (2) $C \vee D \to \sim H$ Rule P

{1,2} (3) $\sim H$ Rule T

{4} (4) $\sim H \to (A \wedge \sim B)$ Rule P

{(1,2),4} (5) $A \wedge \sim B$ Rule T

{6} (6) $(A \wedge \sim B) \to R \vee S$ Rule P

{(1,2),(4),6} (7) $R \vee S$ Rule T

∴ the given argument is valid.

3) If Sachin hits a century then he gets a free car.

Sol: Let p - If Sachin hits a century

q - he gets a free car.

$p \to q$

p

∴ q [Modus Ponens].

4) If I study then I do not fail in the exam. If I donot fail in the exam

→ If I donot fail in the exam then my father gifts a two wheeler to me.

Sol: Let p - If I study

q - I do not fail in the exam

r - my father gifts two wheeler to me.

$p \to q$

$q \to r$

$p \to r$

* theory of Inference for predicate Calculus.

Certain additional rules are required to detail with formulas involving in quantifiers.

→ the elimination of quantifiers can be done by using the rules of specifications called as US and ES.

→ To prefix the correct quantifier we need the rules of generalization called UG & EG.

→ Rule.

* Rule US (Universal Specification):

If a statement of the form $\forall(x)[P(x)]$ is assume to be true then the universal quantifier can be droped to obtain $P(t)$ is true for an object t in the universe.

→ the symbolic form of this rule is

$$\frac{\forall(x)[P(x)]}{P(t) \text{ for all } 't'}$$

* Rule UG (Universal Generalisation):-

If a statement $P(t)$ is true for each element t of the universe then the universal quantifier may be prefixed to obtain $\forall(x)[P(x)]$.

→ the symbolic form of this rule is

$$\frac{P(t) \text{ for all } 't'}{\forall(x)[P(x)]}.$$

* Rule EG (Existential Generalization)

If a statement $P(t)$ is true for some element t in the universe then the existential quantifier may be prefixed to obtain $\exists(x)[P(x)]$ is true.

→ the symbolic form of this rule is

$$\frac{P(t) \text{ for some } 't'}{\exists(x)\ [P(x)]}$$

Example

1) Verify the validity of following argument.

① If every living thing is a plant or an animal.

② John's gold fish is alive and it is not a plant.

③ All animals have heart, therefore john's gold fish has a heart.

Sol: Let us consider

x: Every living thing
P(x): x is a plant
A(x): x is an animal
H(x): x has heart
J(x): John's gold fish

Represent the given statements in symbolic form.

I) Every living thing is a plant or animal.

$\forall(x)[P(x) \vee A(x)]$.

II) John's gold fish is not a plant.

$\sim P(J)$

III.) ~~John~~ All animals have heart.

$\forall(x)[A(x) \rightarrow H(x)]$

IV) John gold fish has a heart.

$H(J)$

$\therefore$ the premises are.

$\forall x[P(x) \vee A(x)]$

$\sim P(J)$

$\forall x[A(x) \rightarrow H(x)]$

$\therefore H(J)$

$\{1\}$	(1)	$\forall x[P(x) \vee A(x)]$	Rule-P
$\{1\}$	(2)	$P(J) \vee A(J)$	Rule-US
$\{3\}$	(3)	$\sim P(J)$	Rule-P
$\{1,3\}$	(4)	$A(J)$	Rule-T (Implication Rule)
$\{5\}$	(5)	$\forall x[A(x) \rightarrow H(x)]$	Rule-P
$\{5\}$	(6)	$A(J) \rightarrow H(J)$	Rule-US
$\{(1,3),5\}$	(7)	$H(J)$	Rule-T ($P, P \rightarrow Q \equiv Q$)

2) Verify the validity of following argument.

1) All men are mortal
2) Socrates is a man.
3) Therefore Socrates is a mortal.

Sol: $H(x)$: x is a man
$M(x)$: x is a mortal
S: S is a Socrates

Symbolic:

$\forall(x)[H(x) \to M(x)]$.

$H(s)$

$M(s)$

∴ the premises are.

$$\forall(x)\,[H(x) \to M(x)].$$
$$\frac{H(s)}{M(s).}$$

{1}	(1)	$\forall(x)[H(x) \to M(x)]$	Rule-P
{1}	(2)	$H(s) \to M(s)$	Rule-US
{3}	(3)	$H(s)$	Rule-P
{1,3}	(4)	$M(s)$	Rule-T

* Predicate Calculus:

The logic based on the analysis of predicate is called predicate logic or predicate calculus.

Note: 1) Connectives can be used & to form the compound statement from the given statement.

2) If S is an n place predicate & $a_1, a_2, a_3, a_4 \ldots a_n$ are the names of the objects then what is the predicate?

$S(a_1, a_2, a_3 \ldots a_n)$ is predicate Statement

→ Compound Statement in predicate Calculus or Logic:

1) Rama is a teacher & her teaching is good

$T(r) \wedge G(t)$

2) Rama is a teacher or her teaching is good

$T \vee F \to T$

$T(r) \vee G(t)$

3) Rama is a teacher then her teaching is good.

$T(r) \to G(t)$.

→ Free variables:

An occurance of a variable that is not bounded by quantifier (either a universal quantifier, Existential quantifier) is said to be a free variable

→ **Bound variables:-**

Or occurance of a variable that is bounded by a quantifier (either universal or existential quantifier) is said to be a bound variable.

→ **Scope of the Quantifiers:-**

→ the part of the logical expression or predicate formula to which a quantifier is applied is called the Scope of the quantifier.

(or)

→ Scope of the quantifier is the formula immediately followed by quantifiers.

	Predicate formula	Bound Variable (B)	Free Variable (F)	Scope of Quantifier (S)
1)	$\forall x\, P(x,y)$	x	y	$P(x,y)$
2)	$\forall x\, [P(x) \to Q(x)]$	x	–	$P(x) \to Q(x)$
3)	$\forall x\, [(P(x) \to \exists(y)\, R(x,y)]$	x, y	–	$\forall x : [P(x) \to \exists(y)\, R(x,y)]$ $\exists y : R(x,y)$
4)	$\forall x\, [P(x) \wedge Q(x)]$	x	–	$P(x) \wedge Q(x)$
5)	$\exists x\, [P(x) \wedge Q(x)]$	x	–	$P(x) \wedge Q(x)$
6)	$\exists x\, P(x) \wedge Q(x)$	first – x	Second – x	$P(x) \wedge Q(x)$

* **Predicates:-**

Predicate is an expression of one or more variables defined on some specified domain

(or)

It is a sentence that contains a finite no. of variables. It becomes a proposition when specific values are substituted for the variables.

→ Predicates has 2 parts in that one is variable / object / Subject & second is predicate.

→ Predicate is denoted with capital letter
a variable is small letter.

Ex: Rama is a bachelor.

$B(x)$.

1) 1-place predicate:-

If there is only 1 object/variable is associated with 1 predicate is called 1-place predicate.

Ex:- x is an even number.

$E(x)$.

2) 2-place-predicate

If there are 2 objects/variables associated with 1 predicate is called 2 place predicate.

Ex: x is greater than y.

$G(x,y)$.

3) n-place predicate:-

If there are n objects/variables is associated with one predicate is called n-place predicate.

Ex: Rama sits between Ravi & Ramu

$S(x, y, z)$.

1) Ex: Let $p(x)$ denotes the statement $x > 3$, what are the truth values of $P(4)$ & $P(2)$

$p(x): x > 3$

$p(4): 4 > 3 \rightarrow T$

$p(2): 2 > 3 \rightarrow F$.

2) Ex: Let $Q(x, y)$ denotes the statement $x = y + 3$. what are the truth values of proposition $Q(1,2)$ & $Q(3,0)$

$x = y + 3$.

$Q(1,2)\ 1 = 2+3$

$1 = 5 \rightarrow F$.

$Q(3,0)\ 3 = 0+3$

$3 = 3 \rightarrow T$.

3) Let $p(x)$, x^2 is a +ve number for every real number x. then what

are truth values of $\forall$.

$P(x) = x^2$

$P(-1) : (-1)^2 = 1 \rightarrow T$

$P(-2) : (-2)^2 = 4 \rightarrow T$

* Quantifiers

Quantifiers are the words that refers to quantities such as Some or all. It tells for how many elements is a given predicate is true.

→ Quantifiers are used to express quantities without giving exact number.

Ex:- all, Some, many, few, none etc.

Ex: Can I have Some water.

It consists of 2 types.

1) Universal Quantifier.

2) Existential Quantifier.

1) Universal Quantifier:

The quantifier 'all' is said to be universal quantifier & it is denoted by $\forall$.

Ex: Let $p(x)$ be a statement then $x+1>x$ what are the truth values for $p(1), p(2)$

$p(x) : x+1 > x$

$p(1) : 1+1 > 1$

$2 > 1$

$p(2) : 2+1 > 2$

$3 > 2$

the truth value is true.

$\therefore p(x)$ is true for all the integers

$\forall x\, p(x)$.

Ex: Every living thing is plant or animal

$P(x)$: x is a plant

$A(x)$: x is a animal

$\forall x\, [P(x) \vee A(x)]$.

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Existential Quantifier:

The quantifier 'Some' is called existential quantifier & it is denoted by $\exists$

Ex: Let $Q(x)$ be a statement then $x < 2$ what are the truth values for $Q(1), Q(2), Q(3)$

$Q(x): x < 2$

$Q(1): 1 < 2$

$Q(2): 2 < 2$

$Q(3): 3 < 2$

$\therefore Q(x)$ is true for only $Q(1) = x = 1$

$\exists x\, Q(x)$.

Q) Some monkeys have tail.

$M(x)$: x is monkey

$T(x)$: x is tail

$\exists x\, [M(x) \rightarrow T(x)]$

1) For the universe of all integers

let $P(x)$: $x > 0$ - +ve integer.

$Q(x)$: x is an even or odd.

$R(x)$: x is perfect square

$S(x)$: x is divisible by 3

$T(x)$: x is divisible by 7.

Write down the following quantifier statements in symbolic form.

i) Atleast one integer is even.

$\exists x\, Q(x)$

ii) There exists a +ve integer & that is even

$\exists x\, [P(x) \wedge Q(x)]$.

iii) Some even integers are divisible by 3.

$\exists x\, (Q(x) \rightarrow S(x)]$.

iv) Every integer is either even or odd

$\forall x\, Q(x)$

v) If x is an even & perfect square then x is not divisible by 3

$\forall x [[Q(x) \vee \sim T(x)] \rightarrow S(x)]$

* logical Equivalence for Quantifiers

two quantifier Statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

Ex: $\forall x\, [p(x) \wedge Q(x)]$

$[\forall x\, p(x)] \wedge [\forall x\, Q(x)]$

Ex: i) $\forall x\, [p(x) \wedge Q(x)] \Leftrightarrow [\forall x\, p(x)] \wedge [\forall x\, Q(x)]$

2) $\exists x\, [p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x)] \vee [\exists x\, q(x)]$

3) $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow [\exists x\, \sim p(x)] \vee [\exists x\, q(x)]$

→ Negation of the quantifier.

Rule for negation of the quantifier is change the universal to existential and vice versa.

Ex: $\sim [\forall x,\, p(x)] \equiv \exists x\, [\sim p(x)]$

$\sim [\exists x,\, p(x)] \equiv \forall x\, [\sim p(x)]$

1) Negation of each given statement
Simplify each of the following given proposition

i) $\exists x\, [\{p(x) \vee q(x)\}] \equiv \forall x\, [\sim p(x) \wedge \sim q(x)]$

ii) $\forall x\, [\{p(x) \wedge \sim q(x)\}] \equiv \exists x\, [\sim p(x) \vee q(x)]$

iii) $\forall x\, [\{p(x) \rightarrow q(x)\}] \equiv \exists x\, [p(x) \wedge \sim q(x)]$

iv) $\exists x\, [\{p(x) \vee q(x)\} \rightarrow r(x)] \equiv$

$\forall x\, [\{p(x) \vee q(x)\} \wedge \sim r(x)]$

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7/11/23 ch:- 2

Set theory

Set is a well defined collection of objects which say

- Sets is denoted by capital letter A, B, C - - - Z
- Element or objects will be denoted by lower cases a, b, c - - - x, y, z
- The phases is belongs to will be denoted by the symbol '$\in$'

Ex:- The states in India
The students who have joined CSE branch in a college

element or member

The object which makes a set are called element or member of the set

Ex:- 1. $x \in A$
x is belongs to A.

2. S = { All even number }
S = { 2, 4, 6, 8 - - - }
$2 \in S$, $4 \in S$,

- set can be represented by using braces

Set Notation :

In generally a set is represent in two ways

1. Roster notation
2. Set builder notation

1 Roaster notation

In reaster notation all the elements are listed and separated by comma and enclosed by braces.

Ex:- 1. Set of all vowels in alphabets
V = { a, e, i, o, u }

2. set of even +ve integer $x \le 10$
E = { 2, 4, 6, 8, 10 }

Let A be a set of letters in the word BOOK
A = { B, O, O, K }

2. Set builder notation

A set builder notation we define the elements of the set by Specifying the characteristics property with all the elements in the set process

Ex:- V = { x : x is a vowel }
E = { x : x is a +ve integer ≤ 10 }

Types of sets :

1. Null set
2. Singletone set
3. Subset
4. Universal set
5. Complement set
6. finite set
7. Infinite set
8. Proper Sub set
9. Equal set
10. Power set

1. A set is said to be null if doesn't consist any elements. example : { }, ϕ

2 A single tone set consists only one element

Ex:- {1}, {2}

3 Sub set :- Let A & B are 2 sets then A is said to be subset of B. If every element of A is an element of B.

$B = \{2, 3, 4, 5, 6, 7\}$

$A = \{3, 6\}$

$A \subseteq B$

4 If all possible sets are subset of a fixed set then a fixed set is called as universal set and denoted by 'U'.

5 Complement set :
set contains all elements of universal set accept set A is called complement of A

Ex:- $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{4, 5, 6\}$

$A^c = \{1, 2, 3\}$

6 Finite set :- If the no of elements in a set are finite or countable then we say a set is a finite set

Ex:-

7 Infinite set : If the no of elements in a set are infinite or non countable then we say that a set is infinite set

Ex:- set ~~or~~ no of stars in sky.

8 If A is subset of B & there is atleast one element of B which is not in A
The symbol subset '⊂' stands for is a proper subset of A & B denoted by

$A \subset B$

Ex:- $A = \{2, 3, 4, 5, 7, 9\}$

$B = \{2, 3, 4, 5, 8, 9, 10, 11\}$

$A \subseteq B = \{2, 3, 4, 5, 7, 9\}$

$A \subset B = \{10, 11\}$

9 Equal set : Two sets A & B are said to be equal if they have precisely 8 elements

Ex:- $A = \{1, 2, 3, 4\}$

$B = \{x / x$ is a positive integer with $x^2 < 20\}$

$\therefore A = B$

note : Two sets A & B are equal. If & only if $A \subseteq B$ & $B \subseteq A$

- For any three set $A \subset B$, $B \subset A$, $C \subset A$

10 Power set : Given a set A, a collection of all subsets of A

- Power set of A is denoted by P(A)
- Suppose set A has n elements the P(A) contains 2^n

Ex:- let $A = \{1, 2, 3\}$ $= 2^3 = 8$

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 2\}$
$\{2, 3\}, \{1, 3\}, \phi\}$

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cartesian product

Let A & B are any two sets the set of all ordered pairs such that the first ele number is element of A & second number is element B denoted is called cartesian product of A & B. can be written as A×B

$A\times B = \{(x,y) / (x\in A) \wedge (y\in B)\}$

Ex:- let $A = \{1,3,5\}$ $C = \{4,6\}$

$B = \{2,3\}$

1. find $A\times B$, $B\times A$, $(A\cup B)\times C$, $(A\times B)\cap C$, $B\times C$, $A\cup B$, $(A\times B)\cap(B\times C)$

$A\times B = \{(1,2),(1,3),(3,2),(3,3),(5,3),(5,2)\}$

$B\times A = \{(2,1),(2,3),(2,5),(3,1),(3,3),(3,5)\}$

$(A\cup B) = \{1,2,3,5\}$

$(A\cup B)\times C = \{(1,4),(1,6),(2,4),(2,6),(3,4),(3,6),(5,4),(5,6)\}$

$B\times C = \{(2,4),(2,6),(3,4),(3,6)\}$

$(A\times B)\cap(B\times C) = \phi$

2. Find x & y in each of the following cases:-

$(2x, x+y) = (6,1)$

$2x = 6$ $x+y=1$

$x = 3$ $3+y=1$

$y = 1-3$

$y = -2$

Relations :-

The relationship between elements of two sets A & B are can are ordered pairs

An ordered pairs commonly known as a point

- If R is a set of ordered pairs i.e (a,b)
- where $a\in A$ & $b\in B$ then R is a relation from A to B
- The relation of set A is on R which is defined as a subset of A×A.

∴ The relation between the set is

$R\subseteq A\times A$

- the relation of set A & B is R which is defined as subset of A×B

$R\subseteq A\times B$

- The relation between a & b is denoted as

$(a,b)\in R$

Ex:- Find out the relation between set A & B on relation R

$A = \{1,2,3\}$, $B = \{1,2,3\}$, $R = \{(1,1),(1,2),(1,3),(2,3)\}$

what is A×B

$A\times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

∴ $R\subseteq A\times B$

⇒ Properties of relations (or) Types of relations :-

A relation has mainly 6 properties

1. Reflexive property
2. Irreflexive property
3. Symmetric

4. Anti symmetric
5. Asymmetric
6. Transitive property

1. Reflexive property / Relation:-
A relation R on a set A is said to be reflexive if $(a,a) \in R$ for every element of $a \in A$.

$\boxed{\text{If } a \in A, \text{ \& } (a,a) \in R \text{ then } R \text{ is Reflexive}}$

Ex:-1 let $A = \{1,2,3\}$
$R = \{(1,1),(2,2),(3,3)\}$
find out reflexive relation of set A on relation R

Sol:- $1 \in A$, $(1,1) \in R$
$2 \in A$, $(2,2) \in R$
$3 \in A$, $(3,3) \in R$
$\therefore$ R is reflexive

2. let $A = \{1,2,3,4\}$
$R = \{(1,1),(2,2),(3,3),(3,4)\}$

Sol:- $1 \in A$, $(1,1) \in R$
$2 \in A$, $(2,2) \in R$
$3 \in A$, $(3,3) \in R$
$4 \in A$, $(3,4) \notin R$
$\therefore$ R is not reflexive

2. Irreflexive Property:-
A relation R on set A is called Irreflexive if $(a,b) \notin R$ for every element of $a \in A$

$\boxed{\text{If } a \in A, \text{ \& } (a,a) \notin R \text{ then } R \text{ is Irreflexive}}$

Ex:- let $A = \{1,2,3\}$
$R = \{(1,1),(1,2),(3,2),(3,3)\}$

Sol:- $1 \in A$, $(1,1) \in R$
$\therefore$ R is not irreflexive

2. let $A = \{1,2,3\}$, $R = \{(1,2),(3,2),(2,3)\}$

Sol:- $1 \in A$, $(1,1) \notin R$
$2 \in A$, $(2,2) \notin R$
$3 \in A$, $(3,3) \notin R$
$(1,2) \in R$, $(3,2) \in R$, $(2,3) \in R$
$\therefore$ It is Irreflexive

3. Symmetric:
A relation R on a set A is called symmetric if whenever $(a,b) \in R$ then $(b,a) \in R$ for all $(a,b) \in A$

$\boxed{\text{If } (a,b) \in A \text{ and } (a,b) \in R \text{ then } (b,a) \in R \text{ then } R \text{ is symmetric}}$

Ex:- let $A = \{1,2,3,4\}$
$R = \{(1,1),(1,2),(2,1),(2,3),(3,2),(1,3)$
$(4,2),(2,4),(3,1)\}$

Sol:- $(1,1) \in R$ & $(1,1) \in R$
$(1,2) \in R$ & $(2,1) \in R$
$(1,3) \in R$ & $(3,1) \in R$
$(2,3) \in R$ & $(3,2) \in R$
$(4,2) \in R$ & $(2,4) \in R$
$\therefore$ R is symmetric

4. Antisymmetric:
A relation R on a set A is called Anti symmetric if whenever $\boxed{(a,b) \in R \text{ \& } (b,a) \in R \text{ then } a=b}$

(or)

A relation R on a set A is called not antisymmetric if whenever $(a,b) \in R$ & $(b,a) \in R$ then $a \neq b$

Ex:- $A = \{1,2,3,4\}$
$R = \{(1,1),(2,2),(3,3),(4,4)\}$

Sol:- $(1,1) \in R$, $(1,1) \in R$ then $1=1$
$(2,2) \in R$ then $2=2$
$(3,3) \in R$ then $3=3$
$(4,4) \in R$ then $4=4$
$\therefore$ R is Antisymmetric

5. Asymmetric
A relation R on a set A is called asymmetric $\boxed{\text{if } (a,b) \in R \text{ implies that } (b,a) \notin R \text{ for all } (a,b) \in A}$

Ex:- $A = \{1,2,3,4\}$
$R = \{(1,2),(3,1),(3,4),(3,2)\}$

Sol:- $(1,2) \in R$ then $(2,1) \notin R$
$(3,1) \in R$ then $(1,3) \notin R$
$(3,4) \in R$ then $(4,3) \notin R$
$(3,2) \in R$ then $(2,3) \notin R$
$\therefore$ R is asymmetric

6. Transitive property:-
A relation R on a set A is called transitive if whenever $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$ for all $(a,b,c) \in A$

Ex:- $A = \{1,2,3,4\}$
$R = \{(1,2),(2,4),(1,4),(3,2),(2,1),(3,1)\}$

Sol:- $[(1,2) \in R, (2,4) \in R, (1,4) \in R]$
$[(3,2) \in R, (2,1) \in R, (3,1) \in R]$
$\therefore$ the R is transitive

⇒ Representation of relations
There are 2 methods used for representing the relations
1. Using Matrix
2. Directed graphs or digraphs

1. Relation matrix
let $A = \{a_1, a_2, a_3 \dots a_m\}$
$B = \{b_1, b_2, b_3 \dots b_n\}$ are finite sets containing m & n elements respectively and R is a relation from A to B

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- Then we can represent a relation R by $m \times n$ matrix called relation matrix
- It is denoted by $M_R = [M_{ij}]$

where $M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$

Ex:- let $A = \{1,2,3\}$

$B = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3)\}$

1. $A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

2. Matrix form $M(A \times B) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

3. If $(a,b) \in A \times B$ & $(a,b) \in R$ then result is 1 otherwise result is '0'.

4. $(1,1) \in (A \times B)$, $(1,1) \in R = 1$

$(1,2) \in (A \times B)$, $(1,2) \notin R = 0$

$(1,3) \in (A \times B)$, $(1,3) \notin R = 0$

$(2,1) \in (A \times B)$, $(2,1) \notin R = 0$

$(2,2) \in (A \times B)$, $(2,2) \in R = 1$

$(2,3) \in (A \times B)$, $(2,3) \notin R = 0$

$(3,1) \in (A \times B)$, $(3,1) \notin R = 0$

$(3,2) \in (A \times B)$, $(3,2) \notin R = 0$

$(3,3) \in (A \times B)$, $(3,3) \in R = 1$

4. Represent the relation in matrix form

$M(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$

5. The complement of relation R is R^C

$R^C = U - R$

$R^C = \{(A \times B) - R\}$

$R^C = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

6. R^C in matrix form

$M[R^C] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

⇒ Relation representation using digraph / directed graph.

A relation can be represented pictorially by drawing its graph.

1. A small circle is drawn from each element of A and marked with each element of A corresponding elements this circle is called vertices.
2. An arrow is drawn from the vertices a_i to a_j if and only if $(a_i, a_j) \in R$ is called as edge.
3. If (a,a) belongs to R then arrow is drawn from a to a such edge is called as loop
4. This pictorial representation of R is called as directed graph

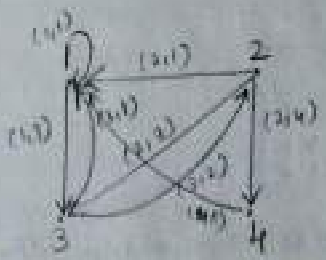
Draw the directed graph of relation

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2)$
$(4,1)\}$ on a the set

$A = \{1,2,3,4\}$

The given relation $R = \{(1,1), (1,3), (2,1), (2,3), (2,4)$
$(3,1), (3,2), (4,1)\}$

The result is



This is digraph of relation R

→ Equivalence Relation

A relation R is said to be equivalence relation if it is reflexive, symmetric & transitive

Ex: Find out whether R is equivalent or not.

$A = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3), (2,3), (3,1)\}$

Reflexive

$1 \in R$, $(1,1) \in R$

$2 \in R$, $(2,2) \in R$

$3 \in R$, $(3,3) \in R$

then R is reflexive

Symmetric :-

$(1,1) \in R$

$(2,2) \in R$

$(3,3) \in R$

$(2,3) \in R$, $(3,2) \notin R$

$(3,1) \in R$, $(1,3) \notin R$

R is not symmetric

∴ R is not equivalent

2. Find out R is equ or not

$A = \{1,2,3,4\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3)$
$(3,2), (4,3), (3,4)\}$

Reflexive

$1 \in A$, $(1,1) \in R$

$2 \in A$, $(2,2) \in R$

$3 \in A$, $(3,3) \in R$

$4 \in A$, $(4,4) \in R$

∴ R is reflexive

Symmetric

$(1,1), (2,2), (3,3), (4,4) \in R$

$(1,2) \in R$, $(2,1) \in R$

$(2,3) \in R$, $(3,2) \in R$

$(4,3) \in R$ $(3,4) \in R$

∴ R is symmetric

Transitive :-

$(1,1) \in R$, $(1,2) \in R$, $(1,2) \in R$

$(1,2)$, $(2,1)$, $(1,1)$

$(2,3)$, $(3,2)$, $(2,2)$

$(4,3)$, $(3,4)$, $(4,4)$

∴ R is transitive

∴ R is equivalent

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⇒ Compatibility relation

An compatibility A relation R is said to be CR if it is reflexive & symmetric

- All equivalence relations are compatibility relation

Ex:- let $A = \{1,2,3,4\}$

$R = \{(1,1),(2,3),(4,3),(3,2),(2,2),(3,4)$
$(3,3),(4,4)\}$

Find whether R is CR or not

Sol:- $1 \in A$, $(1,1) \in R$
$2 \in A$, $(2,2) \in R$
$3 \in A$, $(3,3) \in R$
$4 \in A$, $(4,4) \in R$
$\therefore$ R is reflexive
$(1,1) \in R$, $(2,2) \in R$, $(3,3) \in R$, $(4,4) \in R$
$(2,3) \in R$ & $(3,2) \in R$
$(4,3) \in R$ & $(3,4) \in R$
$\therefore$ R is symmetric
It satisfied 2 conditions
$\therefore$ R is Compatibility Relation

2. let $A = \{1,2,3,4\}$

$R_1 = \{(1,2),(2,3),(3,4),(4,1),(1,1),(2,2),$
$(3,3),(4,4)\}$

$R_2 = \{(3,2),(2,3),(1,1),(2,2),(3,3),(4,4)\}$

Find whether the given relations are CR or not.

$1 \in A$, $(1,1) \in R_1$
$2 \in A$, $(2,2) \in R_1$
$3 \in A$, $(3,3) \in R_1$
$4 \in A$, $(4,4) \in R_1$
$\therefore R_1$ is reflexive
$(2,2),(1,1),(3,3),(4,4) \in R$
$(1,2) \in R_1$ & $(2,1) \notin R_1$
$(3,4) \in R_1$ & $(4,3) \notin R_1$
It is not symmetric
$\therefore R_1$ is not a compatible relation

$1 \in A$, $(1,1) \in R_2$
$2 \in A$, $(2,2) \in R_2$
$3 \in A$, $(3,3) \in R_2$
$4 \in A$, $(4,4) \in R_2$
$\therefore R_2$ is reflexive
$(2,2),(1,1),(3,3),(4,4) \in R_2$
$(3,2) \in R$ & $(2,3) \in R$
$\therefore R_2$ is symmetric
$\therefore R_2$ is a compatibility relation

⇒ Composition of relation:-

let R be a relation from x to y & S be a relation from y to z then the composition relation of R & S is the relation consisting of ordered pairs (x,z) where $x \in X$, $z \in Z$ and for which there exist a element $y \in Y$ such that $(x,y) \in R$ & $(y,z) \in S$.

The elements of CR is

$R : X \to Y \quad x \in X$
$S : Y \to Z \quad y \in Y$
$R \circ S : X \to Z \quad z \in Z$

let $A = \{1,2,3,4\}$ and R & S are 2 relations on set A defined by

$R = \{(1,2),(1,3),(2,4),(4,4)\}$

$S = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\}$

find out RoS, SoR, (RoS)oR, (RoR)oR, RoR, SoS, Ro(SoR)

Sol: RoS = $(1,2)\in R$, $(2,3)\in S$ then $(1,3)$

$(1,2)\in R$, $(2,4)\in S$ then $(1,4)$

RoS $= \{(1,3),(1,4)\}$

SoR = $(1,1)\in S$, $(1,2)\in R$ then $(1,2)$

$(1,1)\in S$, $(1,3)\in R$ then $(1,3)$

$(1,4)\in S$, $(4,4)\in R$ then $(1,4)$

$(2,4)\in S$, $(4,4)\in R$ then $(2,4)$

SoR $= \{(1,2),(1,3),(1,4),(2,4)\}$

(RoS)oR = $(1,4)\in$ RoS, $(4,4)\in R$ then $(1,4)$

(RoS)oR $= \{(1,4)\}$

(RoR) = $(1,2)\in R$, $(2,4)\in R$ then $(1,4)$

$(2,4)\in R$, $(4,4)\in R$ then $(2,4)$

$(4,4)\in R$, $(4,4)\in R$ then $(4,4)$

(RoR) $= \{(1,4),(2,4),(4,4)\}$

(RoR)oR = $(1,4)\in$ RoR, $(4,4)\in R$ then $(1,4)$

$(2,4)\in$ RoR, $(4,4)\in R$ then $(2,4)$

(RoR)oR $= \{(1,4),(2,4)\}$

SoS = $(1,1)\in S$, $(1,1)\in S$ then $(1,1)$

$(1,1)\in S$, $(1,2)\in S$ then $(1,2)$

$(1,1)\in S$, $(1,3)\in S$ then $(1,3)$

$(1,1)\in S$, $(1,4)\in S$ then $(1,4)$

$(1,2)\in S$, $(2,3)\in S$ then $(1,3)$

$(1,2)\in S$, $(2,4)\in S$ then $(1,4)$

SoS $= \{(1,1),(1,2),(1,3),(1,4)\}$

Ro(SoR) = $(1,2)\in R$, $(2,4)\in$ SoR, then $(1,4)$

Ro(SoR) $= \{(1,4)\}$

⇒ Partially ordered set (POSET):

A relation R on a set P is said to be partially ordering relation if and only if R is reflexive, antisymmetric & transitive.

- Partially ordering is denoted by "$\leq$"
- If S a PO on a set P then the order pair $(P, \leq)$ is called a PO set

⇒ Totally ordered relation.

let $(P, \leq)$ is a PO set. if for every 2 elements i.e. $(a,b)\in P$ we have either $a\leq b$, or $b\leq a$ then "$\leq$" is called a simple ordering or linear ordering on P & $(P, \leq)$ is called a totally ordered set

NOTE: 1. $(P, \leq)$ is dual of $(P, \geq)$

2. $(P, \geq)$ is dual of $(P, \leq)$

let $A = \{1,2,3,4\}$ then

$R = \{(1,2),(2,3),(1,1),(1,3),(2,2),(3,3),(4,4)\}$

find whether R is a PO set or not

Sol:- $1 \in A$, $(1,1) \in R$

$2 \in A$, $(2,2) \in R$

$3 \in A$, $(3,3) \in R$

$4 \in A$, $(4,4) \in R$

$\therefore R$ is reflexive

$(1,2) \in R$, $(2,3) \in R$ then $(1,3) \in R$

$(2,3) \in R$, $(3,3) \in R$ then $(2,3) \in R$

$\therefore R$ is transitive

$(1,1) \in R$ $\quad 1=1$

$(2,2) \in R$ $\quad 2=2$

$(3,3) \in R$ $\quad 3=3$

$(4,4) \in R$ $\quad 4=4$

$\therefore R$ is Antisymmetric

$\therefore R$ is PO set

let $A = \{1,2,3,4\}$ then

$R = \{(3,2),(2,3),(1,1),(2,2),(3,4),(4,4)\}$

$(3,3)$

Sol:- $1 \in A$, $(1,1) \in R$

$2 \in A$, $(2,2) \in R$

$3 \in A$, $(3,3) \in R$

$\therefore R$ is reflexive.

11/11/23

⇒ Hasse Diagram:

A pictorial representation of a relation R and all its subsets of relation of R is known as Hash diagrams

- It is a diagram of hash for Poset which doesn't have self loops

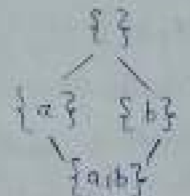
- To simplify the diagram of a partial order i.e Poset we represent the vertices by dots or bullets & draw the diagraph in such a way that all edges point upwards, we need not put arrows in the edge

Condition for Hasse diagram

$(a,b) \in R$ & $a \le b$ or $b \le a$

$R = \{(a,b) \mid a,b \in A$ and "a" divides "b"$\}$

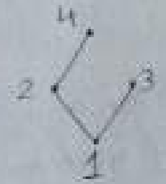
Ex:- $\{a,b\} = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$



1. Draw the hasse diagram for the following diagram & find out relation R

Sol: let $A = \{1,2,3,4\}$

$R = \{(1,2),(1,3),(1,4),(2,4)\}$

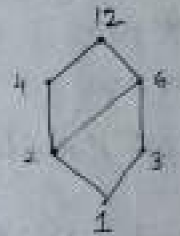


2 let $A = \{1,2,3,4,6,12\}$ on A
define the relation R by aRb if and only if a divides b. Prove that R is poset on A and draw hasse diagram for Relation R

sol: $2^6 = 64$

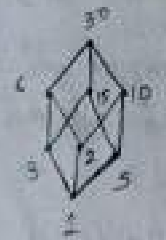
R = { 64 elements }

Prove A is POSET



3 let $A = \{1,2,3,5,6,10,15,30\}$

$2^8 = 256$



⟹ functions:-

A mapping f i.e $A \to B$ is set to be function if every element of A having unique image in B. If A, B are two sets then A is called as domain & B is range for that domain.

Representation of function:-

$f: A \to B$

A f B

let A & B are two sets. A function f i.e A to B is a relation from A to B such that for every element in set A there is an element or range in set B

A f B

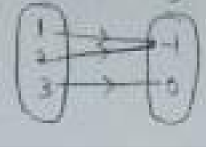
Here y is called image of x and x is called the preimage of y under function f

1 let $A = \{1,2,3\}$ and $B = \{-1,0\}$, S be a relation from A to B defined by $S = \{(1,-1),(2,-1),(3,0)\}$. Is S is a function or not and find the domain & codomain of S

sol Given sets are $A = \{1,2,3\}$

$B = \{-1,0\}$

A f B



∴ S is a function i.e $S: A \to B$

Domain $A = \{1,2,3\}$

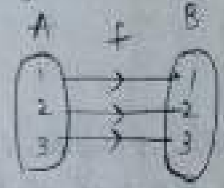
Co-domain $B = \{-1,0\}$

Types of function:-

1. Identity
2. constant
3. one - one function
4. Onto function
5. Bijective

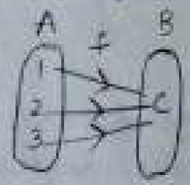
1 Identity:- If a function $f(x)$ is said to be identity function if $f(x) = x$,

Ex:-



2 Constant: If a function $f(x)$ is said to be constant if $f(x) = c$

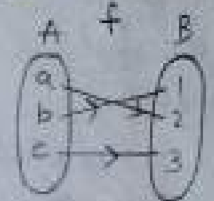
Ex:-



$f : A \to B$

3 one - one function:-

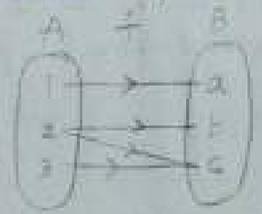
A function $f(x)$ is said to be one - one fun if every element of A must have distinct image in B



$f(a) = 2$
$f(b) = 1$
$f(c) = 3$

4 Onto function :-

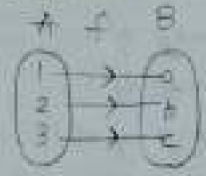
A fun $f(x)$ is said to be onto fun if every element of B must have pre image in A



'1' is the pre image of a
'2' is the pre image of b, c
'3' is the pre image of c

5. Bijective:-

A function $f(x)$ is said to be bijective if the function is one - one & onto

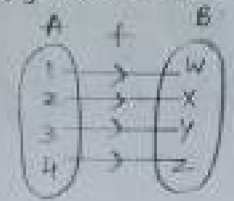


$f : A \to B$

1 is the pre - image of a
2 is the pre - image of b
3 is the pre - image of c

Functions:

If $A = \{1, 2, 3, 4\}$, $B = \{W, X, Y, Z\}$ & $F = \{(1, W), (2, X), (3, Y), (4, Z)\}$ Then f is an one - one and onto function.



$f : A \to B$

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2. If function $F: N \to N$ and $f(x) = x^2+1$ then check whether F is a bijective or not

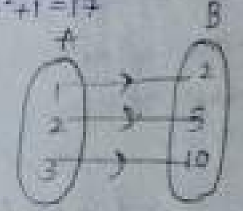
The given function $F: N \to N$, where N is a natural number i.e. $N = \{1, 2, 3 \dots\}$

$f(1) = 1^2+1 = 2$

$f(2) = 2^2+1 = 5$

$f(3) = 3^2+1 = 10$

$f(4) = 4^2+1 = 17$



$f: A \to B$

Therefore, the above function is bijective, because

1 is the pre-image of 2

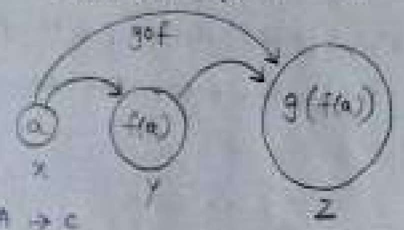
2 is the pre-image of 5

3 is the pre-image of 10

From mapping and pre images, the function is bijective.

Composite functions:

Let us consider $f: A \to B$, $g: B \to C$ then the composite function is defined as $gof: A \to C$



$gof: A \to C$

$g(f(a)) = f(a) = a$

Ex: 1 let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $C = \{w, x, y, z\}$ with $f: A \to B$ & $g: B \to C$ given by function

$F = \{(1, a), (2, a), (3, b), (4, c)\}$ & $g = \{(a, x), (b, y), (c, z)\}$ Find out gof

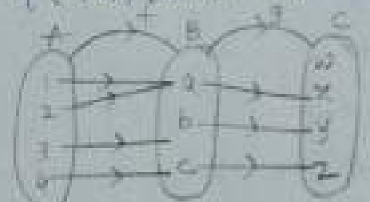
$gof: g(f(a)) = g(f(1)) = g(a) = x$

$= g(f(2)) = g(a) = x$

$= g(f(3)) = g(b) = y$

$= g(f(4)) = g(c) = z$

$gof = \{(1, x), (2, x), (3, y), (4, z)\}$



2 let $f(x) = x+1$

$g(x) = x^2$

and find whether $fog = gof$

Then find $fog = gof$ and find whether $fog = gof$

$fog = fog(x) = f(x^2) = x^2+1 \to ①$

$gof = gof(x) = g(x+1) = (x+1)^2 \to ②$

$\therefore$ from eqn ① & ② $fog \neq gof$

3 let $f(x) = x+5$ & $g(x) = x^2+1$ Then find fog & gof and whether fog is equal to gof or not

$fog = fog(x)$

$= f(x^2+1)$

$= x^2+1+5 = x^2+6$

$gof = gof(x)$

$= g(x+5)$

$= (x+5)^2+1$

$= x^2+10x+26$

4 Let $f(x)=x^2$, $g(x)=x+1$, & $h(x)=x$, then find fogoh, gofoh, gohof and fohog

Sol: 1 $fogoh = fogoh(x)$

$= fog(x)$

$= f(x+1)$

$= (x+1)^2$

$gofoh = gofoh(x)$

$= gof(x)$

$= g(x^2)$

$= x^2+1$

$gohof = gohof(x)$

$= goh(x^2)$

$= g(x^2)$

$= x^2+1$

$fohog = fohog(x)$

$= foh(x+1)$

$= f(x+1)$

$= (x+1)^2$

⇒ Inverse function:-

A function $f: A \to B$ is said to be invertible if the relation is $f^{-1}: B \to A$ is also function then f^{-1} is called Inverse of f.

A function is said to be inverse it must be one-one and onto function.

- steps to follow the finding inverse of a function

1 Replace $f(x)$ with y

2 Interchange x's with y's

3 solve y

4 replace y with $f^{-1}(x)$

Problems:-

1 Let $f(x)=x^2$, then find $f^{-1}(x)$

Let $f(x)=y$

$x = f^{-1}(y)$

$f(x)=x^2$

$y = x^2$

$x = \sqrt{y}$

$f^{-1}(y) = \sqrt{y}$

$f^{-1}(x) = \sqrt{x}$

2 Find the inverse of the following $f(x) = \frac{3x+2}{2x+1}$

Let $f(x)=y \Rightarrow x = f^{-1}(y)$

Given, $f(x) = \frac{3x+2}{2x+1}$

$y = \frac{3x+2}{2x+1}$

Replace x with y & y with x

$x = \frac{3y+2}{2y+1}$

$2xy + x = 3y+2$

$2xy - 3y = 2-x$

$y(2x-3) = 2-x$

$y = \frac{2-x}{2x-3}$

$\therefore f^{-1}(x) = \frac{2-x}{2x-3}$

3 let $f(x) = x+1$. Find $f^{-1}(x)$

let $f(x) = x+1$

$f(x) = y$

$y = x+1$

$x = y+1$ (interchange)

$y = x-1 \quad \therefore f^{-1}(x) = x-1$

4 let $f(x) = x+5$ and $g(x) = x^2+1$ then find $f^{-1}og^{-1}$ and $g^{-1}of^{-1}$

$f(x) = x+5, \ g(x) = x^2+1$

let, $f(x) = y$

$x+5 = y$

$y+5 = x$ (interchange)

$y = x-5$

$f^{-1}(x) = x-5$

$g(x) = x^2+1$

let $g(x) = y^1$

$y^1 = x^2+1$

$x = (y^1)^2+1$

$x-1 = (y^1)^2$

$\sqrt{x-1} = y^1$

$\rightarrow g^{-1}(x) = \sqrt{x-1}$

$f^{-1}og^{-1} = f^{-1}og^{-1}(x)$

$= f^{-1}(\sqrt{x-1})$

$= \sqrt{x-1} - 5$

$g^{-1}of^{-1} = g^{-1}of^{-1}(x)$

$= g^{-1}(x-5)$

$= \sqrt{x-5-1} = \sqrt{x-6}$

Recursive Function:

A function f on a argument x is generally specified by indicating the value of $f(x)$ for every value of x in the domain of f in the explicit form

- Sometimes it is possible to define a function in terms of itself. This process is called Recursive function
- Apply a rule to its result is $f(x) = n!$ here n is every natural number

$f(0) = 1$ and $f(n) = 1, 2, 3 \ldots n$ for $n \in Z^+$

- The recursive method of describing this function is $f(0) = 1$ and $f(n) = nf(n-1)$ for $n \in Z^+$
- The recursive function for square of n is

$f(0) = 0$

$f(n) = f(n-1) + 2n-1$

$f(1) = f(1-1) + 2(1) - 1 = 1 = 1^2$

$f(2) = f(2-1) + 2(2) - 1 = f(1)+4-1 = 1+4-1 = 4 = 2^2$

$f(3) = f(3-1) + 2(3) - 1 = f(2)+6-1 = 4+6-1 = 9 = 3^2$

- For factorial $f(n) = n$ is defined recursively by

$f(0) = 1$

$f(n) = nf(n-1)$

$f(1) = 1 f(1-1)$

$= 1 f(0) = 1.1 = 1$

- $f(2) = 2 f(2-1)$

$= 2 f(1) = 2.1 = 2$

- $f(3) = 3 f(3-1)$

$= 3 f(2) = 3.2 = 6$

- $f(4) = 4 \cdot f(4-1)$

$= 4 f(3) = 4.6 = 24$

Ex:-
Recursive definition for the function $f(x) = a_n$, in each of the following cases

i) $a_n = 5n$
$a_0 = 5(0)$
$a_0 = 0$
$a_1 = 5(1) = 5$

ii) $a_n = 6^n$
$a_0 = 6^0 = 1$
$a_1 = 6^1 = 6$

5/12/23 CH:3 Algebraic Structure

Algebraic structure:-
The algebraic structure is a type of non empty set S which is mapped with one or more than one binary operation

$N = \{1, 2, 3 \ldots \infty\}$
N is a set of all natural numbers.

$Z = \{0, \pm 1, \pm 2, \ldots \infty\}$
is the set of all integers

Q is the set of all rational numbers

R is the set of all real numbers

Binary Operation:-
Let A be a non empty set then "*" is said to be a binary operation if it is closed under that operation

- closed means perform the operations on the set and the result is also belongs to that set is called closed

Say $N = \{1, 2, 3 \ldots \infty\}$
$(N, +)$ is a binary operation
Take the addition of $2 + 3 = 5$

$(N, *)$ is a binary operation
Take the multiplication of $6 * 3 = 18$
18 belongs to N

$(N, -)$ it is not a binary operation
$4 - 6 = -2$ doesn't belong to set N

UNIT-3

Algebraic Structures

Syllabus:

* Introduction
* Algebraic Systems
* Semigroups & Monoids.
* Lattices as Partially Ordered Sets.
* Boolean Algebra.

Algebraic Systems:

A system consisting of a non empty set & 1 or more n array operations on the set is called as algebraic system.

→ An algebraic System will be denoted by $\{S, f_1, f_2 \ldots f_n\}$.

where S is the non empty set & $f_1, f_2 \ldots f_n$ are the n array operations.

→ We will mostly deal with algebraic System with number $N = \{0, 1 \& 2\}$ containing one or two operations only.

* **Types of Algebraic Structures / System :-**

there are various types:

1. Semi group
2. Monoids
3. Group
4. Abelian group.

① Semi Group:-

Let S be a non-empty set & * be a binary operation on S then algebraic system <S, *> is called a semi group if & only if it satisfies the following properties.

i) closure.

ii) Associative.

Eg:- Prove that $(E, *)$ is Semi group.

take, Suppose $E = \{2, 4, 6, 8 \ldots\}$.

$(a, b) \in E$

$a = 2, b = 4$, ~~[illegible]~~

$a * b = 2 * 4 = 8 \in E$

$\therefore$ closure property for $(a, b) \in E$.

$\therefore$ $(E, *)$ is Semi group.

Associative property.

$a * (b * c) = (a * b) * c$

$2 * (4 * 6) = (2 * 4) * 6$

$2 * 24 = 8 * 6$

$48 = 48$

$a,b,c \in E$

$\therefore$ It is associative property.

$\therefore (E, *)$ is Semigroup.

② Monoid:

Let S be a non empty S.

'+' be a binary operation on S.

then algebraic system $<S, +>$ is called a monoid. If it satisfies the following properties.

i) closure

ii) Associative

iii) Identity.

Ex:-

① $<W, +>$ is a monoid. where W = set of whole numbers i.e., $W = \{0, 1, 2, \ldots \infty\}$.

② $<N, +>$ is a monoid, N is all natural numbers. i.e., $N = \{1, 2, 3 \ldots \infty\}$.

$<N, +>$ is not a monoid because '0' is not there is set of Natural numbers.

Note:- A monoid is always a semigroup

⑧ Groups:-

Let S be a non-empty set & + be a binary operation on S then the algebraic system $<S,+>$ is called group. It should satisfy the properties.

i) closure.
ii) Associative
iii) Identity
iv) Inverse

A group is always a semigroup & monoid.

In group G it has only one inverse. Let E be a identity element in a. Suppose a' and a'' are inverse of any element a i.e $a \in G$

$$a' = a'e$$
$$= a'(aa'') \quad [\because e = aa'']$$
$$= (a'a)a'' \quad [\because \text{Associative property}]$$
$$= 1 \cdot a''$$
$$= a''$$

(1) Abelian Group:-

Let S be a non-empty set & * be a binary operation on S, then the algebraic system on S.

<S, +> or <S, *> is called a Abelian group if it satisfies with the following properties.

i) closure

ii) Associative

iii) Identity.

iv) Inverse.

v) Commutative.

Ex:- Prove that G = {1, -1, i, -i}.

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Above all entities belong to G. So it satisfies closure property.

let $a=1, b=-1, c=i$

we have, $a*(b*c) = (a*b)*c$

$$1*(-1*i) = (1*(-1))*i$$

$$1*(-i) = (-1)*i$$

$$\boxed{-i = -i}$$

$\therefore$ ~~It~~ satisfies Associative.

$\therefore$ Multiplication is associative on complex no.

$\therefore$ $(G, *)$ is associative.

$$\boxed{a*e = e*a = a}$$

→ $1*1=1$

→ $-1*1=-1$

→ $i*1=i$

→ $-i*1=-i$

$\therefore$ All the outputs are belongs to G

$\therefore$ $(G, *)$ is satisfies Identity property on G

$a * a^{-1} = e$

$\rightarrow 1 * 1^{-1} = 1$

$\rightarrow -1 * -1^{-1} = 1$

$\rightarrow i * i^{-1} = 1$

$\rightarrow -i * -i^{-1} = 1$

$\therefore$ $(G, *)$ Satisfies Inverse property.

$a * b = b * a$.

$1 * (-1) = (-1) * 1$

$-1 = -1$

$1 * i = i * 1$

$i = i$

$\therefore$ $(G, *)$ satisfies Commutative property

$\therefore$ Given set G satisfies all properties.

$\therefore$ the given set is a Abelian Group.

* Lattices:-

A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements such as $a, b \in L$, has a greatest lower bound (GLB) & a least upper bound (LUB)

→ the LUB (Supremum) of a subset $a, b \subseteq L$ is denoted by $a \vee b$ (or) $a \oplus b$ & is called the join or sum of a, b.

→ the GLB (Infimum) of a sub set $a, b \subseteq L$ is denoted by $a \wedge b$ (or) meet (or) $a * b$ and product of a and b

Partial order

A relation R on set A is said to be a partial ordering relation (or) a partial order on A if

i) R is reflexive

ii) anti symmetric

iii) transitive on A.

→ A set A with partial order R defined on it is called a partially ordered set (or) an ordered set (or) a poset, and is denoted by the pair (A, R)

$(Z, \leq)$ is a poset.

$(Z, \geq)$ is a poset.

$(Z, <)$ is not poset.

$(Z, >)$ is not poset.

Ex: $A = \{1, 2, 3\}$.

$R = \{(1,1) (1,2) (1,3) (2,2) (2,3) (3,3)\}$.

Reflexive → $(1,1) (2,2) (3,3)$ → ✓

Anti Symmetric → $(1,2) \in R, (2,1) \notin R$ → ✓

$(1,3) \in R, (3,1) \notin R$ → ✓

Transitive → $(1,1)(1,2) \in R$ $(1,2)$ ✓

∴ $(Z, \leq)$ is a poset.

Hasse diagram

the digraph of poset is called Hasse diagram (or) poset diagram.

Rules for Hasse diagram.

1) No loops for Reflexive.
2) No Edge for transitive Relation
3) If $(a,b) \in R$ there must be an upward edge from a to b.
4) down to up.

Example:-

1) If R is a relation on the set $A = \{1, 2, 3, 4\}$ defined by xRy if x divides y. Prove that (A, R) is a poset. Draw its Hasse diagram.

$R = \{(x, y) \mid x, y \in A$ and x divides y$\}$

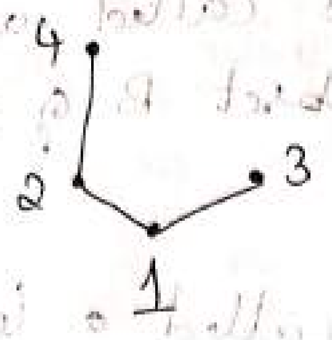
Sol: $= \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)$, $(4,4)\}$.

$\Rightarrow \{(1,1)\ (2,2)\ (3,3)\ (4,4)\} \in R. \rightarrow$ Reflexive.

$\Rightarrow \{(1,2) \in R$ but $(2,1) \notin R$ $\rightarrow$ Antisymmetric
$(1,3) \in R$ but $(3,1) \notin R$
$(2,4) \in R$ but $(4,2) \notin R\}$

$\Rightarrow \{(1,2)\ (2,4) \Rightarrow (1,4) \in R\} \rightarrow$ Transitive.

So (A, R) is a poset.



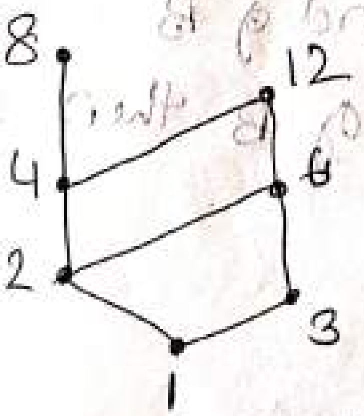
a) Let $A = \{1, 2, 3, 4, 6, 8, 12\}$ on A, define the partial ordering Relation R by $a R b$ if and only if $a | b$. Draw the Hasse diagram for R.

b) Write down the relation matrix for R.

Sol: $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,6)\ (1,8)\ (1,12)$
$(2,2)\ (2,4), (2,6)\ (2,8)\ (2,12),$
$(3,3)\ (3,6)\ (3,12)\ (4,4)\ (4,8)\ (4,12)$
$(6,6)\ (6,12)\ (8,8)\ (12,12)\}.$

The Hasse diagram for this R is as shown below.

$(1,2)\ (2,4) \Rightarrow (1,4)$
$(1,3)\ (3,6) \Rightarrow (1,6)$
$(1,6)\ (6,12) \Rightarrow (1,12)$
$(1,2)\ (2,4) \Rightarrow (1,4)$
$(1,4)\ (4,8) \Rightarrow (1,8)$



*Rules for LUB & GLB:

→ An element $a \in A$ is called an upper bound of a subset B of A if $xRa \; \forall x \in B$.

→ An element $a \in A$ is called a lower bound of a subset B of A if $aRx \; \forall x \in B$

→ An element $a \in A$ is called a least upper bound (LUB) of a subset B of A if the following two conditions holds:

i) a is an upper bound of B
ii) If a' is an upper bound of B then aRa'

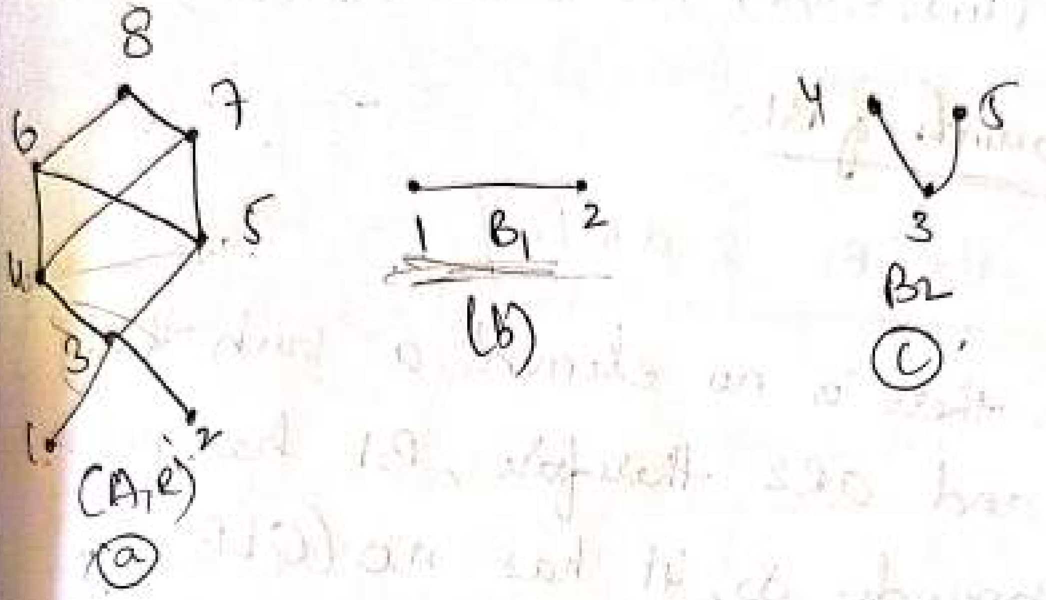
→ An element $a \in A$ is called a greatest lower bound (GLB)

i) a is a lower bound of B
ii) if a' is a lower of B then $a'Ra$

→ Example:-

Let $A = \{1,2,3,4,5,6,7,8\}$ and a partial order on A whose Hasse diagram given below & consider the subsets.

$B1 = \{1,2\}$ $B_2 = \{3,4,5\}$ of A.



① Upper bounds of B1 = (3, 4, 5, 6, 7, 8).

→ In this '3' is the least upper bound of B1.

Condition for upper bound of B1

$\boxed{a \in A \quad xRa \ \forall x \in B1.}$

1 R3
2 R3 ⇒ 3 is the upper bound.

1 R4
2 R4 ⇒ 4 is the upper bound.

1 R5
2 R5 → 5 " "

1 R6
2 R6 → 6 " "

1 R7
2 R7 → 7 " "

1 R8
2 R8 → 8 " "

$$\boxed{\underset{B}{(1,2)}\ R\ (\overset{a}{3}, \underbrace{4,5,6,7,8}_{a'})}$$

∴ a = Selected upper
a' = remaining upper

$\boxed{a\ R\ a'}$

if 3R(4,5,6,7,8) the 3 is LUB (B1)

② Lower bounds of B1 :-

$\boxed{aRx \quad \forall x \in B1 \ \& \ a \in A}$

→ In A, there is no element a such that aR1 and aR2 therefore, B1 has no lower bounds. So, it has no (GLB).

① Upper bound of B2 :- (6, 7, 8).

$\boxed{xRa \quad \forall x \in B_2 \ \& \ a \in A}$

3R6
4R6 ⇒ 6 is upper bound.
5R6

3R7
4R7 ⇒ 7 is upper bound.
5R7

3R8
4R8 ⇒ 8 is the upper bound.
5R8

3 R 4
4 R 4 ⇒ So 4 is not upper bound.
5 R 4 ✗

3 R 5 ✓
4 R 5 ✗ ⇒ So 5 is not upper bound.
5 R 5 ✓

3 R 3 ✓
4 R 3 ✗ ⇒ So, 3 is not upper bound.
5 R 3 ✗
[basically lower]

$\underbrace{(3, 4, 5)}_{B}$ R (6, 7, 8).

6, 7, 8 upper bound of B_2 of A.

LUB ⇒ aRa'

6 R 7 ✗
6 R 8 ⇒ 6 is not LUB So, no LUB of B

7 R 6 ✗
7 R 8 ⇒ 7 is not LUB.

Lower bound of B_2

$\boxed{x R B_2}$

3 R 3 ✓
3 R 4 ✓ ⇒ 3 is the lower bound
3 R 5 ✓

1 R 3
1 R 4 ⇒ 1 is the lower bound
1 R 5

2 R 3
2 R 4 ⇒ 2 is lower bound.
2 R 5

GLB is '3',

a'	a
1 R 3	
2 R 3	

Problems :-

1)

*	a	b	c	d
a	a	c	b	d
b	c	b	d	a
c	b	d	a	c
d	d	a	c	b

On the set $A = \{a, b, c, d\}$. a binary operation $*$ is defined as described in the above table. is Commutative ? Associative.

Sol: Commutative but not Associative.

2) In each of the following cases a binary operation $*$ on set $A = \{a, b\}$ is defined through a multiplication table. Determine whether $\langle A, * \rangle$ is a semigroup or a monoid or neither.

①

*	a	b
a	b	a
b	a	b

②

*	a	b
a	a	a
b	b	b

③

*	a	b
a	a	b
b	a	a

Problems :-

1)

*	a	b	c	d
a	a	c	b	d
b	c	b	d	a
c	b	d	a	c
d	d	a	c	b

On the set $A = \{a, b, c, d\}$. a binary operation * is defined as described in the above table. is Commutative ? Associative

Sol: Commutative but not Associative.

2) In each of the following cases a binary operation * on set $A = \{a, b\}$ is defined through a multiplication table. Determine whether $\langle A, * \rangle$ is a semigroup or a monoid or neither.

①

*	a	b
a	b	a
b	a	b

②

*	a	b
a	a	a
b	b	b

③

*	a	b
a	a	b
b	a	a

Sol: Since $*$ is a binary operation on A,
we have to verify only associative
law & the existence of identity if any.

i) we find that:

$a*(a*a) = a*b = a = b*a = (a*a)*a$

$a*(a*b) = a*a = b = b*b = (a*a)*b$

$a*(b*a) = a*a = (a*b)*a$

$a*(b*b) = a*b = (a*b)*b$

$b*(a*a) = b*b = b = a*a = (b*a)*a$

$b*(a*b) = b*a = a = a*b = (b*a)*b$

$b*(b*a) = b*a = (b*b)*a$

$b*(b*b) = b*b = (b*b)*b$.

These relations show that $*$ is associative.
we further note that:

$a*b = a, \ b*a = a, \ b*b = b.$

Thus, b is the identity element in A
under $*$. Hence, A forms a monoid
under $*$. We check that this is
commutative also.

ii) As in the above case, we check that the associative law holds. But neither a nor b is the identity element, bcoz, $a*b=a$ & $b*a=b$. Since, $a*b \neq b*a$, Commutative law does not hold.

→ Hence $\langle A, * \rangle$ is a non commutative semigroup. it is not a monoid.

iii) we find that.

$$b*(a*b) = b*b = a$$
$$(b*a)*b = a*b = b.$$

these show that $*$ is not associative. therefore, A does not form a semigroup under $*$. It cannot be a monoid.

(OR).

Sol: i)

*	a	b
a	b	a
b	a	b

Associative :-

$$\boxed{(a*b)*c = a*(b*c)}$$

$(a*a)*a = a*(a*a)$. $(a*b)*c = a*(b*c)$

$b*a = a*b$. $a*a = a*a$

$a = a$. $a = a$.

$(a*a)*b = a*(a*b)$. $(a*b)*c = a*(b*c)$

$b*b = a*a$. $a*b = a*b$

$b = b$. $a = a$

$\therefore$ It satisfies associative property

$\boxed{\text{Identity :- } e*a = a*e = a}$

$a*e = e*a$.

$a = a$

$b*e = e*b$

$b = b$.

$\therefore$ ~~It~~ All the outputs belong to A.

$\therefore$ $(A,*)$ Satisfies Identity property.

$\therefore$ It is monoid.

ii)

$*$	a	b
a	a	a
b	b	b

$A = \{a, b\}$.

$a = a,\ b = a,\ c = a$

$a = a,\ b = a,\ c = b$

$$\boxed{a * (b * c) = (a * b) * c}$$

$a * (a * a) = (a * a) * a$

$a * a = a * a$

$a = a$ //

$a * (a * b) = (a * a) * b$

$a * a = a * b$

$a = a$ //

$a * b = a * b$

$a = a$ //

$b * (a * b) = (b * a) * b$

$b * a = b * b$

$b = b$ //

$\therefore$ It is Associative

$$\boxed{\text{Identity} \Rightarrow a * e = e * a = a}$$

$a * e = e * a$

$a * a = a * a$

$a = a$

$a * e = e * a$

$a * b = b * a$

$a \neq b$

$\therefore$ It is not Identity.

$\therefore$ It is not monoid.

iii)

$*$	a	b
a	a	b
b	a	a

$a*(a*a) = (a*a)*a$

$a*a = a*a$

$a = a$

$a*(a*b) = (a*a)*b$

$a*b = a*b$

$b = b$ //

∴ It is associative.

$\boxed{a*e = e*a}$

$a*a = a*a$

$a = a$

$a*b = b*a$

$b \neq a$

∴ It is not identity

∴ It is not monoid

$a*a)*b$

$a*b = a*b$

$b = b$ //

$b*(a*b) = (b*a)*b$

$(b*b) = a*b$

$a \neq b$

It is not associative

Q) To S.T the set Q of all rational numbers forms a commutative monoid under the operation $*$ defined by

$a*b = a+b-ab$,

We need to prove two properties.

1. Associative: $(a*b)*c = a*(b*c)$.
2. Identity: $a*e = e*a = a$.

1. Associative:

$$a*b = a+b-ab$$

Now compute $(a*b)*c$:

$$\Rightarrow (a*b)*c = (a+b-ab)*c$$
$$= (a+b-ab)+c-(a+b-ab)c$$
$$= a+b-ab+c-ac-bc+abc.$$

Next compute $a*(b*c)$:

$$\boxed{b*c = b+c-bc}$$

$$a*(b*c) = a*(b+c-bc)$$
$$= a+(b+c-bc)-a(b+c-bc)$$
$$= a+b+c-bc-ab-ac+abc.$$

$\therefore$ We see that both expressions simplify to.

$$a+b+c-ab-ac-bc+abc.$$

thus, $(a*b)*c = a*(b*c)$.

$\therefore$ It is associative.

2) Identity:

Set $a*e = a$.

$\Rightarrow a*e = a+e-ae = a.$

Solving for e:

$$a+e-ae = a.$$
$$e-ae = a-a$$
$$e-ae = 0.$$
$$e(1-a) = 0.$$

for e to satisfy this equation for all $a \in \mathbb{Q}$, e must be 0. thus, the identity element e is 0.

Verify this:-

$a * 0 = a + 0 - a \cdot 0$

$= a$

$0 * a = 0 + a - 0 \cdot a$

$= a$

③ Commutative

finally, we need to show that the operation $*$ is commutative.

$a * b = a + b - ab$

$b * a = b + a - ba$.

Since addition & multiplication of rational numbers are commutative,

$a + b - ab = b + a - ba$

thus, $a * b = b * a$.

∴ We have shown that the operation $*$ is associative, commutative, & has an identity element 0

∴ $(\mathbb{Q}, *)$ forms a commutative monoid.

5) **Ring**: A set R equipped with two binary operations + (addition) and × (multiplication) such that

→ (R, +) is an abelian group.

→ (R, ×) is a semigroup.

→ Distributive properties:

$a \times (b+c) = (a \times b) + (a \times c)$

$(a+b) \times c = (a \times c) + (b \times c)$ for all

$a, b, c \in R$.

Example problem:

1) If N denotes the set of all natural numbers, and + and × are the usual addition and multiplication operations show that $\langle N, +, \times \rangle$ is not a ring.

Sol: Given,

To prove that $\langle N, +, \times \rangle$ (the set of natural numbers with usual addition & multiplication) is not a ring, we need to verify if it satisfies the properties of a ring.

→ A ring must satisfy several properties, including:

1) $(N, +)$ forms an abelian (commutative) group.

2) $(N, \times)$ forms a semigroup (associative multiplication).

3) Distributive properties:

$$a \times (b+c) = (a \times b) + (a \times c)$$

$$(a+b) \times c = (a \times c) + (b \times c).$$

1) $(N, +)$ must form abelian group.

→ It must satisfy.

(i) closure under addition.

(ii) associativity of addition.

(iii) Existence of an additive identity.

(iv) Existance of additive inverse.

(v) Commutativity of addition

Natural numbers N (typically defined as $\{1, 2, 3 \ldots\}$ or $\{0, 1, 2, \ldots\}$) are closed under addition & addition is associative & commutative. The additive Identity is 0 when we include it in the set of natural numbers.

→ However, natural numbers do not have additive inverse within the set of natural numbers.

→ For instance, the additive inverse of 1 is -1, which is not a natural number. Hence, (N, +) does not form an abelian group.

2) (N, x) forms a semigroup.

To be a semigroup, the set must be closed under multiplication & multiplication must be associative. Natural numbers are closed under multiplication & multiplication is associative. Hence, (N, x) is a semigroup.

3) Distributive properties:

~~We need to check if the distributive~~

It has to satisfy the distributive properties.

$a \times (b+c) = (a \times b) + (a \times c)$

$(a+b) \times c = (a \times c) + (b \times c)$.

these properties hold true for natural number because of the standard properties of addition & multiplication.

∴ the Key issue is that $(N,+)$ does not form an abelian group because natural numbers lack additive inverses within the set of natural numbers.

therefore, $\langle N,+,\times\rangle$ does not satisfy all the necessary properties to be a ring.

thus $\langle N,+,\times\rangle$ is not a ring.

Hence, proved.

Problem.

Let G be the set of all non-zero real numbers and let $a*b = \frac{1}{2}ab$. Show that $\langle G, * \rangle$ is an abelian group.

Sol: for any two non-zero real numbers a & b, we note that $\frac{1}{2}(ab)$ is a non-zero real number.

To prove $\langle G, * \rangle$ is an abelian then, it has to satisfy following properties.

① Closure: for all $a, b \in G$,
$$a * b \in G.$$

② Associative: for all $a, b, c \in G$,
$$(a*b)*c = a*(b*c).$$

③ Identity element: there exists an element $e \in G$. Such that
$$a*e = e*a = a.$$
for all $a \in G$.

④ Inverse element: for every $a \in G$, there exists an element $a' \in G$ such that $a*a' = a'*a = e$.

(5) <u>Commutative</u> :- for all $a, b \in G$, ~~a~~

$$a * b = b * a.$$

(1) $a * b = \frac{1}{2} ab$.

→ Since, a & b are non-zero real numbers their product ab is also a non-zero real number.

→ Dividing a non-zero real number by 2 still results in a non-zero real number. thus, $a * b \in G$.

→ (2) first compute $(a*b)*c$.

$(a*b)*c = \underline{\left(\frac{1}{2} ab\right) * c}$ → we are assuming ~~a*c~~

~~$a*c = \frac{1}{2} ac$~~

~~$\Rightarrow \frac{1}{2} ab * \frac{1}{2} ac$~~

~~$\frac{1}{2}\left(\frac{1}{2} ab\right)c$~~

Multiplying $\frac{1}{2}$ → $= \frac{1}{2}\left(\frac{1}{2} ab\right)c$

$\Rightarrow \frac{1}{2} \cdot \frac{1}{2} \cdot abc$

$= \frac{1}{4} abc.$

→ next, compute $a * (b*c)$. ∵ $\boxed{b*c = \frac{1}{2} bc}$

$a*(b*c) = a * \left(\frac{1}{2} bc\right)$.

$= \frac{1}{2} a \left(\frac{1}{2} bc\right)$

$= \frac{1}{4} abc.$

Since $\frac{1}{4} abc = \frac{1}{4} abc$, we have satisfied associative property.

3) Identity-

Assume e is the identity element.

then, $a * e = \frac{1}{2}ae = a$.

Solving for e : $\frac{1}{2}ae = a$.

$\frac{1}{2}e = \frac{\cancel{a}}{\cancel{a}}$

$\frac{1}{2}e = 1$

$e = 1 \times 2$

$\boxed{e = 2}$.

let's verify this.

$a * 2 = \frac{1}{\cancel{2}}a \cdot \cancel{2} = a$.

$2 * a = \frac{1}{\cancel{2}} \cdot \cancel{2}a = a$.

$\therefore$ 2 is the identity element for $*$.

4) Inverse : $a * a' = e = 2$

$a * a' = \frac{1}{2}aa' = 2$

Solving for a' :

$\frac{1}{2}aa' = 2$

$aa' = 2 \times 2$

$aa' = 4$

$\boxed{a' = \frac{4}{a}}$

Since $a \neq 0$, $a' = \frac{4}{a}$ is also a non-zero real number and belongs to G.

Verify this: $\boxed{a * a' = \frac{1}{2} aa' = 2}$

$$a * \left(\frac{4}{a}\right) = \frac{1}{\cancel{2}} \cancel{a} \cdot \frac{\cancel{4}^{2}}{\cancel{a}}$$

$$= 2.$$

$$\left(\frac{4}{a}\right) * a = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\cancel{a}} \cdot \cancel{a}$$

$$= 2.$$

$\therefore$ the inverse of a is $\frac{4}{a}$.

5) Commutative: $\rightarrow a * b = b * a$.

$a * b = \frac{1}{2} ab$

$b * a = \frac{1}{2} ba$

Since multiplication of real numbers is commutative, $ab = ba$.

Therefore, $\frac{1}{2} ab = \frac{1}{2} ba$

$a * b = b * a$. //

$\therefore$ It is commutative.

Therefore, $\langle G, * \rangle$ is an abelian group.

It satisfies every property in abelian group.

(6) Boolean Algebra.

Definition: A lattice that also supports complementation and satisfies additional properties.

Properties:

→ Binary operations: AND, OR, & NOT

→ Identities for AND and OR, distributive complement laws.

* Axioms or properties of Boolean Algebra.

If $a, b, c, \in B$ then

Commutative law

(1) $a \cdot b = b \cdot a$

(2) $a + b = b + a$

Distributive law

(1) $a + (b \cdot c) = (a+b) \cdot (a+c)$

(2) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

Identity law

(1) $a + 0 = a$

(2) $a \cdot 1 = a$

Complement Law :-

① $a + \bar{a} = 1$

② $a \cdot \bar{a} = 0$

Binary Operation:-

Let A be a non empty set then "*" is said to be a binary operation if it is closed under that operation.

- closed means perform the operations on the set and the result is also belongs to that set is called closed.

Say $N = \{1, 2, 3, \ldots \infty\}$

$(N, +)$ is a binary operation

Take the addition of $2+3 = 5$

$(N, *)$ is a binary operation

Take the multiplication of $6*3 = 18$

18 belongs to N

$(N, -)$ it is not a binary operation

$4-6 = -2$ doesn't belong to set N.

• If any two numbers perform the operation if result is also belong to that set then it is called Binary operation.

→ Properties of Binary operation:-

1. closure property:
Take any 2 no. from the set and perform the operation. The result must belong to same set called closure property.

• when the operation satisfies closure property it should be a binary operation otherwise not a BO

2. Associative property:-
$a*(b*c) = (a*b)*c$

3. Identity property:-
$a*e = a$, e is Identity element
$e*a = a$

Ex:- $2+0=2$ $2*1=2$
$3+0=3$ $3*1=3$

4. Inverse Element:-
$a*a^{-1}=e$

Ex:- $2+(-2)=0$, $2*2^{-1}=1$
$6+(-6)=0$ $7*7^{-1}=1$

5. Commutative property
$a*b=b*a$
Let * be a binary operation on S. The operation * is said to be commutative in S if
$a*(b*c) = (a*b)*c$

6. Distributive property
For any element a, b, c belongs to S then
$a*(b+c) = (a*b)+(a*c)$

7. Idempotent property:-
An element a belongs to S is called an idempotent element w.r. to the operation *, +
$a*a=a$

Ex:- $a \in z$ is an idempotent element under addition operation.
$0+0=0$
$1*1=1$

Unit - IV

Elementary Combinatorics.

Basics of Counting:

There are two basic rules of counting

1. Sum Rule
2. product Rule.

Sum Rule: If a task can be done in n_1 ways and a second task in n_2 ways and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Ex: The department will award a free computer to either a cs student or a cs professor.

Sol: How many different choices are there, if there are 530 students and 15 professors?

- there are $530 + 15 = 545$ choices.

Ex: suppose there are 16 boys and 18 girls in a class How many ways to select one of these students either a boy or a girl as class representative?

no.of boys = 16

no.of girls = 18

Task-1: no.of ways for selecting a boys as a class representative

= m-ways = 16 ways

Task-2: no.of ways for selecting a girl as a class representative

n = ways = 18 ways

no.of ways of selecting a boy (or) a girl as a class representative = $16 + 18 = n + m$

= 34 ways.

product rule: suppose that a procedure can be broken down into a sequence of two tasks

- if there are n_1 ways to do 1^{st} task and for each of these ways of doing the 1^{st} task, there are n_2 ways to do the procedure

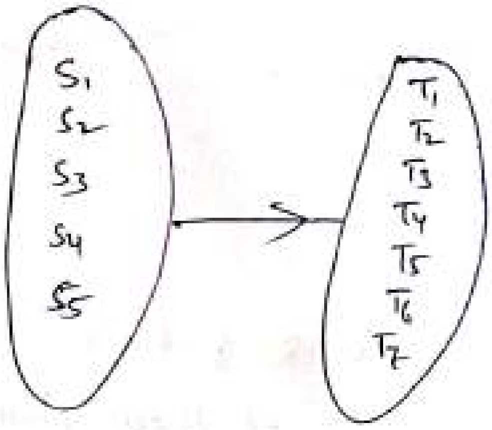
Ex: How many different license plates are there that containing exactly ~~three~~ three english letters?

Sol: there are 26 possibilities to pic the first letter; then 26 possibilities for the second one and 26 for the last one.

$\therefore$ so there are $26 \cdot 26 \cdot 26 = 17576$ diff.. licence plate.

Ex: Suppose a person has 5 shirts and 7 ties How many ways a person can choose a shirt and Tie?

Sol:



n - shirts
M - ties.

$S_1 \rightarrow$ Among 7 ties he can choose one tie. in 7 ways.

$S_2 \Rightarrow$ 7 ways

$S_3 \rightarrow$ 7 - ways

$S_4 \rightarrow$ 7 - ways

$S_5 \rightarrow$ 7 - ways

$= M \times n$

$= 7 \times 5 \Rightarrow 35$ ways.

permutation and combination

permutations An Arrangement in a sequence of elements of a set is called permutation of elements.

$$^nP_r = \frac{n!}{(n-r)!}$$ - no.of ways to arrange the elements

Ex: ABC - ABC, ACB, BAC, BCA, CAB, CBA } 6 ways.

sum rule

How many ways can we selected girls or boys representatives from the class of 30 girls, 40 boys

$= 30 + 40$

product rule

How many ways can we selected girls and boys representatives from the class of 30 girls & 40 boys

$= 30 \times 40$

Ex: 7 letters can be arranged in 4 spaces.

A, B, C, D, E, F, G - n.

_ _ _ _ - r.

$$^nP_r = \frac{n!}{(n-r)!} = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}}$$

$=$

$1! = 1$

$2! = 2$

$3! = 3\times2 = 6$

$4! = 4\times3\times2\times1 = 24$

$5! = 5\times4\times3\times2\times1 = 120$

$6! = 6\times5\times4\times3\times2\times1 = 720$

$7! = 7\times5\times4\times3\times2\times1 = 5040$

Combination: An ordered selection of n elements of a set containing n distinct element is called an n-combinations of n elements and is denoted by $c(n,r)$ or n_{c_r} or n_r

$$n_{c_r} = \frac{n!}{(n-r)!\,r!}$$

Example problems on permutations.

Ex: How many ~~ways~~ different strings of length 4 can be formed using the letters of the word "PROBLEM"?

the given word is "PROBLEM" has 7 letters

$\therefore n = 7$

The no. of different strings of length 4 can be formed by using the letters of the word "PROBLEM" is $np_r = \frac{n!}{(n-r)!} = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}}$

$= 840$ ways.

(2) Find the no. of permutations of the letter of the word ENGINEERING?

the given word is ENGINEERING has 11 letters.

out of this letters 3 are E,

3 are N

2 are I

2 are G

1 is R.

∴ the total no. of permutations of the letters of word ENGINEERING is

$$= \frac{n!}{n_1! \times n_2! \times n_3! \times n_4! \times n_5!} = \frac{11!}{3! \times 3! \times 2! \times 2! \times 1!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times \cancel{6} \times 5 \times \cancel{4} \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2} \times 1 \times \cancel{3 \times 2 \times 1} \times \cancel{2 \times 1 \times 2 \times 1 \times 1}} = 11 \times 10 \times 9 \times 8 \times 7 \times 5$$

$= 2,70,200$ ways.

Ex. How many 4 digit no. can be formed by using the digits 2, 4, 6, 8 when repetation of digit is allowed.

$\boxed{4}$ $\boxed{4}$ $\boxed{4}$ $\boxed{4}$ 2 4 6 8

thousand hundreds tens unit

$= 4 \times 4 \times 4 \times 4$

Total how many 4 digit no. can be write $= 4 \times 4 \times 4 \times 4$
$= 256$ ways.

Ex. How many ways 4 digits Even no. can be formed by using digits 1, 2, 3, 4, 6, 8 when repetation of digit is allowed.

given digits are 1, ~~2~~, 3, 4, 6, 8 = 5 digits

even num = 4, 6, 8 - 3 ways.

→ even no. Unit can be selected in 3 ways

$= 5\ \ 5\ \ \boxed{5}\ \ 5\ \ \boxed{3}$ places.

thou Hundreds unit

$= 5 \times 5 \times 5 \times 5 \times 3$

$= 375$ ways.

Ex: How many 4 digit no. can be formed from six digits ~~&~~ repetation is not allowed (6)

1, 2, 3, 5, 7, 8

$n = 6$

Sol: Repetation ~~are~~ not allowed. $r = 4$

given digits - 1, 2, 3, 5, 7, 8.

for digits number $= {}^{n}P_{r} = {}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!}$

$= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} = 360$ ways

Ex: How many of such number is less than 4000 ? the digits are 1, 2, 3, 5, 7, 8. repetation is not allowed

$\underline{3}$ $\underline{5}$ $\underline{4}$ $\underline{3}$ $\boxed{1, 2, 3}$, 5, ~~4~~, 7, 8

Thousand Hund ten Unit

$= 3 \times 5 \times 4 \times 3 =$

1000 place containing 3 ways that is 1, 2, 3.

∴ the total no. of less than 4000 is.

$= 3 \times 5 \times 4 \times 3$

$= 180$.

Combinations Example problems: Selecting the elements

$$n_{C_r} = \frac{n!}{(n-r)!\,r!}$$

Ex ① $10_{C_2} = \frac{10!}{(10-2)!\,2!} = \frac{10!}{8!\,2!} = \frac{10 \times 9 \times \cancel{8!}}{2!} = \frac{\cancel{10}^{5} \times 9}{\cancel{2}} = 45$

Ex ② Find the value of 'n' if $n_{C_{n-2}} = 10$.

$r = n-2$

$$n_{C_r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{[n-(n-2)]!\,(n-2)!} = 10 \Rightarrow \frac{n!}{(\cancel{n}-\cancel{n}+2)!\,(n-2)!} = 10 \Rightarrow \frac{n!}{2!\,(n-2)!} = 10$$

$\therefore 9_{C_2} = \frac{9 \times 8 \times 7!}{1 \times 2}$ like $n! = (n-1)!\,n$

$$= \frac{n(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}} = 10$$

$n^2 - n = 20$

$n^2 - n - 20 = 0$

$n^2 - 5n + 4n - 20 = 0$

$n(n-5) + 4(n-5) = 0 \Rightarrow (n-5)(n+4) = 0$

$n-5 = 0$ $\qquad$ $n+4 = 0$

$\boxed{n=5}$ ✓ $\qquad$ $\boxed{n=-4}$ → in the combination no need to take negative value

$\therefore$ n - value is 5.

Ex: A committee of 5 people is to be formed a group of 4 men and 7 women How many possible committes can be formed if at least 3 women are in the committee

Total numbers of committe is = 5 (8)

Men = 4

Women = 7

$\boxed{\begin{array}{l} nc_1 = n \\ nc_0 = 1 \end{array}}$

No. of committes $= {}^7C_3\,{}^4C_2 + {}^7C_4\,{}^4C_1 + {}^7C_5\,{}^4C_0$

$= \frac{7\times6\times5}{3\times2\times1}\cdot\frac{4\times3}{2\times1} + \frac{7\times6\times5\times4}{4\times3\times2\times1}\cdot\frac{4}{1} + \frac{7\times6\times5}{\ldots}\cdot 1$

$= 35\cdot6 + 35\cdot4 + 21$

$= 210 + 140 + 21$

$= 371$

$\boxed{\begin{array}{l} nc_1 = n \\ nc_0 = 1 \end{array}}$

Ex: How many automative license plates can be made if each plate contains two diff. letters ~~formed~~ followed by 3 different digits. solve the problem if the first digit can not be zero.

letters = 26.

digits - 3 non - zero.

1^{st} possition $= 26c_1$

2^{nd} possition $= 25c_1$

□□□ (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

1 - 2nd digit 8

1st digit - (non-zero) $= 9c_1$

2nd digit - $9c_1$

3rd digit - $8c_1$

Total no of license plates are $= 26c_1 \times 25c_1 \times 9c_1 \times 9c_1 \times 8c_1$

$= 26\times25\times9\times9\times8$

$= 421200$

permutation with Repetition:

If p represents no. of different permutations of n things taken all at a time, when p of them are of one kind, q are of them another kind, r of third kind and so on, then

$$p = \frac{n!}{p!\,q!\,r!}$$

Ex: How many arrangements can be made with letters of the word EXAMINATION.

sol. given word is EXAMINATION.

Total letters in the given word is = 11

$$p = \frac{n!}{p!\,q!\,r!}$$

repeated letters are A = 2
I = 2
N = 2

$$p = \frac{11!}{2!\,2!\,2!} = \frac{11\times10\times9\times8\times7\times6\times5\times\cancel{4}\times3\times\cancel{2\times1}}{\cancel{2\times1}\times\cancel{2\times1}\times\cancel{2\times1}}$$

$$= 11\times10\times9\times8\times7\times6\times5$$

$$= 49{,}89{,}600$$

Ex: INDEPENDENCE

Total letters = 12 $\quad p = \frac{n!}{p!\,q!\,r!} = \frac{12!}{3!\,3!\,3!}$

N = 3
D = 3
E = 3

$12\times11\times10\times9\times8$

Case-i : when there is no restrictions

Case-ii :- when all the words begin (or) ends with perticular ~~le~~ letters

Case-iii :- when few letters / vowels & consonents occur Together

Case-iv :- when vowels consonents occupy odd / even place

$$\text{Total no.of ways} = \frac{n!}{x!y!z!}$$

Ex: Find the no.of permutations that can be made out of the letters

Case-i

MISSISSIPPI

Total letters - 11

M occurs = 1

'I' repeated in 4 times

'S' " " 4 Times

'P' " " 2 "

$$\text{Total no.of permutations} = \frac{11!}{1!\,4!\,4!\,2!}$$

$$= \frac{11 \times 10 \times 9 \times \cancel{8}^{4} \times 7 \times 6 \times 5 \times \cancel{4!}}{1!\,\cancel{2!}\,\cancel{4!}\,4!} = 34654$$

case-iii :- All 'S' together

$\boxed{SSSS}$ — 1 digit - MIIIPPI - 7 digits

7 + 1 = 8 digits. M - 1 time
I - 4 time
P - 2 time

$$\text{Total no of permu} = \frac{8!}{4!\,2!\,1!} = \frac{\overset{4}{\cancel{8}}\times 7\times 6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4!\times 2!\times 1!}}$$

$$= 4\times 7\times 6\times 5$$

$$= 840 \text{ ways}$$

Find the no. of words that can be formed by ~~using~~ using all the letters of the word "RAINBOW"

(a) How many of them begins with R and ends with W.

(b) How many of them ends with BOW

(c) " " " begin with ROW

Sol: The word RAINBOW has 7 letters

$\therefore$ Total no. of word $= n! = 7! = 5040$

(a) the word begin with 'R' and end with W.

$$\underline{R}\underbrace{AINBO}_{5!\text{ ways}}\underline{W}$$

$5!$ ways $= 5\times 4\times 3\times 2\times 1$
$= 180$ ways.

(b) ~~the~~ the word ends with $\underline{BOW}$

$$\underbrace{RAIN}_{4!\text{ ways}}\underline{BOW}$$

Remaining 4 letter can be arranged 4! ways — $4\times 3\times 2\times 1 = 24$ ways

(c) the word begin with ROW

$\underbrace{\text{RAINBOW}}_{4!\text{ ways}}$ —— ROW

the Remaining '4' letters can be arranged in

4 ways that is $4! = 24$ ways

Ex: (3) In how many ways can the letters of word "LAUGHTER" be arranged so that vowels are always together

Sol: the given word is LAUGHTER

total letters of the word is = 8

Vovels are = 3 - AUE - 1 word.

consonants are = 5

$\underset{1}{\underline{AUE}}\ \underset{2}{L}\ \underset{3}{G}\ \underset{4}{H}\ \underset{5}{T}\ \underset{6}{R}$

Total can be arrenged in 6 ! ways

For Each of this 3 vovels can be arranged in 3! ways.

∴ total word $= 6! \times 3!$

$= 720 \times 6$

$= 4320$

Ex ① In How Many ways can the letter of the word HEXAGON be arranged so that the ~~vod~~ vowels are always in even place (13)

Sol: the word HEXAGON has 7 letters of which '3' are vowels

since total letters is 7, there are 4 odd & 3 even places

The 3 vowels can be arranged in the 3 odd place in 3! ways

for each of this 4 consonants can be arranged in 4 odd places in 4! ways

Hence total ways $= 3! \times 4!$

$= 6 \times 24$

$= 144$ ways

Binomial Theorem:

A binomial theorem describes the algebraic expression of powers of a binomial with 2 variables.

$$(x+y)^n = {}^nC_0 \cdot x^n y^0 + {}^nC_1 \cdot x^{n-1} y^1 + {}^nC_2 \cdot x^{n-2} y^2 + \cdots + {}^nC_{n-1} \cdot x^1 \cdot y^{n-1} + {}^nC_n \, x^0 \, y^n$$

it can be written as

$$\boxed{(x+y)^n = \sum_{r=0}^{\infty} {}^nC_r \, x^{n-r} \, y^r}$$

$$(1+x)^n = {}^nC_0 \, 1^n \cdot x^0 + {}^nC_1 \, 1^{n-1} \cdot x^1 + {}^nC_2 \, 1^{n-2} \cdot x^2 + {}^nC_3 \cdot 1^{n-3} \cdot x^3 + \cdots$$

$$= {}^nC_0 \cdot x^0 + {}^nC_1 \cdot x^1 + {}^nC_2 \, x^2 + {}^nC_3 \, x^3 + \cdots$$

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$$

$$= 1 \cdot x^0 + \frac{n!}{(n-1)!\,1!} \cdot x + \frac{n!}{(n-2)!\,2!} \cdot x^2 + \frac{n!}{(n-3)!\,3!} \cdot x^3 + \cdots$$

$$= 1 + \frac{n\cancel{(n-1)}}{\cancel{(n-1)}!\,1!} \cdot x + \frac{n\,(n-1)\cancel{(n-2)}}{\cancel{(n-2)}!\,2!} \cdot x^2 + \frac{n(n-1)(n-2)\cancel{(n-3)}}{\cancel{(n-3)}!\,3!} \cdot x^3$$

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \cdots}$$

Ex: find out the coefficient of $x^9 . y^3$ in the expension of $(x+2y)^{12}$ (15)

Sol: we know that Binomial theorem

$$(x+y)^n = \sum_{r=0}^{\infty} n_{c_r} \cdot x^{n-r} \cdot y^r$$

According to the binomial theorem

$x = x, \quad y = 2y, \quad n = 12$

$$(x+2y)^{12} = \sum_{r=0}^{\infty} 12_{c_r} \cdot x^{12-r} (2y)^r$$

$$= \sum_{r=0}^{\infty} 12_{c_r} \cdot x^{12-r} \cdot 2^r \cdot y^r$$

$$= \sum_{r=0}^{\infty} 12_{c_r} \cdot 2^r \cdot x^{12-r} \cdot y^r \quad \text{--- (1)}$$

we have to find out $x^9 . y^3$

x & y terms can be compare with equ-(1) is

$$x^9 y^3 = x^{12-r} \cdot y^r$$

$x^9 = x^{12-r}$

$9 = 12 - r$

$r = 12 - 9$

$\boxed{r = 3}$

$y^3 = y^r$

$\boxed{3 = r}$

substitute $r=3$ in equ-(1) is

$$= \sum_{3=0}^{\infty} 12_{c_3} \cdot 2^3 \cdot x^{12-3} \cdot y^3$$

$$= 12_{c_3} \cdot 2^3 \cdot x^9 \cdot y^3$$

$$= 12_{c_3} \cdot 2^3 \Rightarrow \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220 \times 2^3 = 1760$$

$\therefore$ the coefficient of $x^9 y^3$ in $(x+2y)^{12}$ is 1760

Ex: Find out the coefficient of x^5y^2 in the expression of $(2x-3y)^7$ $\boxed{6048 - \text{Ans.}}$ (16)

Q: Find out the coefficient of x^5 in the expansion of $(1-2x)^{-7}$

power value is -ve - formula

Sol: we know that $\boxed{(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \cdot x^r}$

According to the above formula, $x = 2x$, $n = 7$

$\boxed{x = 2x}$ & $\boxed{n = 7}$

Substituting this values in the above formula

$$(1-2x)^{-7} = \sum_{r=0}^{\infty} {}^{7+r-1}C_r \cdot (2x)^r$$

$$= \sum_{r=0}^{\infty} {}^{6+r}C_r \cdot 2^r \cdot x^r \quad \text{--- (1)}$$

We have to find out the coefficient of x^5

$x^r = x^5$

$\boxed{r = 5}$

$$= \sum_{r=0}^{\infty} {}^{6+r}C_r \cdot 2^r \cdot x^r$$

$$= {}^{6+5}C_5 \cdot 2^5 \cdot x^5$$

$$= {}^{11}C_5 \cdot 2^5 \cdot x^5$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times \cancel{6}}{\cancel{5 \times 4 \times 3 \times 2 \times 1}} \cdot 2^5 = 14784 \cdot x^5$$

∴ the coefficient of x^5 in the expression of $(1-2x)^{-7}$ is 14784

Ex: Find the coefficient of x^5 in the expression of $(1-2x)^{-7}$

(17)

Sol: - power is negative value -7 = the formula is

$$\boxed{(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \cdot x^r}$$

According to the above formula

$\cancel{-}x = \cancel{-}2x$ | $\cancel{-}n = \cancel{-}7$

$\boxed{x = 2x}$ | $\boxed{n=7}$

Substituting this values in above formula is

$$(1-2x)^{-7} = \sum_{r=0}^{\infty} {}^{+7+r-1}C_r (2x)^r$$

$$= \sum_{r=0}^{\infty} {}^{6+r}C_r \cdot 2^r \cdot x^r \quad \text{—(1)}$$

We have to find out the coefficient of x^5.

$$x^r = x^5$$

$$\boxed{r=5}$$

r value substituted in equation-(1)

$$= \sum_{r=0}^{\infty} {}^{6+5}C_5 \, 2^5 \cdot x^5$$

$$= {}^{11}C_5 \cdot 2^5 \cdot x^5$$

$$= 14784\, x^5$$

∴ The coefficient of x^5 in the expression of $(1-2x)^{-7}$ is 14784

Ex: Find out the coefficient of x^{27} in the expansion of $(x^4+x^5+x^6+\dots)^5$

x not posible

$-(x\pm y)^n$ or $(1+x)^n$ ↳ x-terms.

Sol: Given that $(x^4+x^5+x^6\dots)^5$ (18)

$= [x^4(1+x+x^2\dots)]^5$

$= (x^4)^5 (1+x+x^2\dots)^5$

$= x^{20}\cdot(1+x+x^2\dots)^5$

$= x^{20}((1-x)^{-1})^5$ $\qquad [\because (1-x)^{-1} = 1+x+x^2+\dots]$

$= x^{20}(1-x)^{-5}$ — (1)

we know that

$$\boxed{(1-x)^{-n} = \sum_{r=0}^{n} {}^{n+r-1}C_r\cdot x^r}$$

$1-x = 1-x$ | $-n = -5$

$\boxed{x=x}$ | $\boxed{n=5}$

$$(1-x)^{-5} = \sum_{r=0}^{\infty} {}^{5+r-1}C_r\cdot x^r$$

$$= \sum_{r=0}^{\infty} {}^{4+r}C_r\cdot x^r \quad \text{— (1)}$$

$$x^{20}(1-x)^{-5} = x^{20}\cdot\sum_{r=0}^{\infty} {}^{4+r}C_r\cdot x^r$$

$$= \sum_{r=0}^{\infty} {}^{4+r}C_r\cdot x^{20}\cdot x^r$$

$$= \sum_{r=0}^{\infty} {}^{4+r}C_r\, x^{20+r} \quad \text{— (2)}$$

we have to find out the coefficient of x^{27}

(19)

$x^{27} = x^{20+r}$

$27 = 20 + r$

$\boxed{r = 7}$

$r = 7$ is substituted in equation-2

$= \sum_{r=0}^{\infty} {}^{4+7}C_7 \cdot x^{20+7}$

$= {}^{11}C_7 \cdot x^{27}$

$= 330\ x^{27}$.

$\therefore$ the coefficient of x^{27} is 330.

Multinomial Theorem:

Multinomial theorem is a generalization of the binomial with more than two variables.

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1, n_2, \ldots n_k} \frac{n!}{n_1! * n_2! * \cdots * n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

Ex: Compute the following by using multinomial theorem.

(a) $\binom{7}{2,3,2}$ (b) $\binom{4}{1\ 1\ 2}$

(a) Ans: it is in the form of $\binom{n}{n_1, n_2, n_3}$ where

$$n_1 + n_2 + n_3 = n$$

by Applying the multinomial theorem

$$= \frac{n!}{n_1! * n_2! * n_3!} = \frac{7!}{2! * 3! * 2!} = \frac{7 \times \cancel{6}^{2} \times 5 \times 4 \times 3 \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{3} \times \cancel{2} \times 1 \times 2 \times 1}$$

$$= 210$$

(b) According to multinomial theorem

$$= \frac{4!}{1! * 1! * 2!} = \frac{4 \times 3 \times \cancel{2} \times 1}{1 \times 1 \times \cancel{2} \times 1} = 12$$

Ex: Determine the coefficient of (1) xyz^2 in the expension of $(2x-y-z)^4$ (2-1)

Sol: by applying the multinomial theorem

$$(x_1+x_2+\cdots x_k)^n = \frac{n!}{n_1! * n_2! * \cdots * n_k!} \cdot (x_1)^{n_1} * (x_2)^{n_2} * \cdots (x_k)^{n_k}.$$

where $n_1+n_2+\cdots+n_k = n$

The given expression is $(2x-y-z)^4$

where $n=4$, $x_1 = 2x$, $x_2 = -y$, $x_3 = -z$

Apply multinomial theorem

$$\frac{n!}{n_1! * n_2! * n_3!} \cdot (x_1)^{n_1} * (x_2)^{n_2} * (x_3)^{n_3} = \frac{4!}{n_1! * n_2! * n_3!} * (2x)^{n_1} * (-y)^{n_2} * (-z)^{n_3} \rightarrow (1)$$

we have to find out the coefficient of xyz^2 is

$xyz^2 = x_1^{n_1} * x_2^{n_2} * x_3^{n_3}$ $\boxed{x^1y^1z^2 = n_1, n_2, n_3}$

where $n_1 = 1$, $n_2 = 1$, $n_2 = 2$

Substituting n_1, n_2, n_3 values in equation-(1)

$$= \frac{4!}{1! * 1! * 2!} (2x)^1 * (-y)^1 * (-z)^2$$

$$= \frac{4\times3\times\cancel{2}\times1}{1!\times1!\times\cancel{2}\times1} 2x^1 \cdot (-y)^1 \cdot z^2 \Rightarrow 12 \cdot 2x (-y) \cdot z^2$$

$$= -24\, xyz^2$$

$\boxed{xyz^2 = -24}$

Example problems on combinations with repetition

① In how many ways can we distribute 12 Identical pencils to 5 childrens so that every child gets atleast one pencil?

no. of pencils = 12

no. of childrens = 5

every child gets atleast one pencil

Means each child may get ≥ 1 pencil

1st we distribute one pencil to each children then the remaining Identical pencils ~~(12-5)~~ (12-5=7) can be distributed to 5 children

no. of ways to distribute 7 identical pencils to 5 childrens is = $C(r+n-1, r)$

where $r=7$
$n=5$

$= C(7+5-1, 7)$

$= C(11, 7)$

$= \frac{11!}{7!(11-7)!} = \frac{11!}{7! * 4!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times \ldots}{7 \times 6 \times 5 \times 4 \times 3 \ldots \times 1}$

$= 330$ ways

Example problems on permutation. (23)

① Find the no. of permutations of the letters of the word MASSASAUGA. . I. how many of these all four A's together ?

II. How many of them begins with S ?

Sol: the given word "MASSASAUGA" has 10 letters $\therefore n = 10$

out of 10 letters 4 are A
3 are S
1 are M
1 are U
1 are G

$\therefore$ the required no. of permutations

$$= \frac{n!}{n_1! * n_2! * n_3! * n_4!}$$

$$= \frac{10!}{4! * 3! * 1! * 1! * 1!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1} \times 3 \times 2 \times 1 \times 1 \times 1 \times 1}$$

$= 25,200$ ways.

I. In a permutation. all A's are together. we treat all of A's as one letter. then the letters to be permutated read (AAAA), S,S,S, M, U, G and the no of permutations are

$$= \frac{10!}{1! \times 3! \times 4! \times 1! \times 1! \times 1!} = 840$$

ii. for permutation begining with S there occur

9 open partitions to fill. where 2 are S
4 " A &
1 each are M, U, G

the no. of such permutations are

$$= \frac{9!}{2! \times 4! \times 1! \times 1! \times 1!}$$

MASSASAUGA
S _ _ _ _ _ _ _ _ _ (1 2 3 4 5 6 7 8 9 10) = 9! = 10-1

= 7560

Example problems of combinations with repetitions.

→ suppose we wish to select a combinations of "r" objects with repetition from a set of "n" distinct objects. the no. of such selections is given by

n - distinct objects
↓
r - Identical objects with repetitions
∴ No. of such selections are
$C(n+r-1, r) = (r+n-1, n-1)$

$$C(n+r-1, r) = C(r+n-1, n-1)$$

→ the following are the other interpretations of this no. is

(i) $C(n+r-1, r) = C(r+n-1, n-1)$ repetitions the no. of ways in which "r" identical objects can be distributed among n - distinct ~~combinations~~ containers

(ii) $c(n+r-1, r) = c(r+n-1, n-1)$ repetition the no. of non-negative integer solutions of the equation.

<u>Note</u>: A non-negative Integers solution of the equation $x_1 + x_2 + \dots + x_n = r$ is an n-tuple, where $x_1, x_2, x_3 \dots x_n$ are non-negative Integers where sum is 'r'.

<u>Ex</u>: In how many ways can we distribute 10 Identical marbles among 6 distinct container?

<u>Sol</u>: 10 Identical marbles is $r = 10$

6 distinct containers are $n = 6$

number of such selections $= C(r+n-1, r)$

$$= C(10+6-1, 10)$$

$$= C(15, 10)$$

$$= \frac{15!}{10! \times 5!} = \frac{n!}{(n-r)!\, r!} = \frac{15!}{(15-10)!\, 10!}$$

$$= \frac{\cancel{15}^{\,3} \times \cancel{14}^{\,7} \times 13 \times \cancel{12}^{\,3} \times 11 \times \cancel{10!}}{\cancel{10!} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 91 \times 33$$

$$= 3003$$

Ex: Find the number of non-negative Integers ~~selections~~ solutions of the equation $x_1+x_2+x_3+x_4+x_5=8$

Sol: x_1, x_2, x_3, x_4, x_5 are the non-negative Integers

sum of this Integers are $= 8 = r$

$n = x_1, x_2, x_3, x_4, x_5 = 5$

$n = 5 =$ (no. of non-negative Integers) $= 5$

$r = 8$

$$= C(n+r-1, r)$$

$$= C(5+8-1, 8)$$

$$= C(12, 8) \qquad \boxed{= \frac{n!}{(n-r)!\,r!}}$$

$$= \frac{12!}{(12-8)!\,8!} = \frac{12 \times 11 \times 10 \times 9 \times \cancel{8!}}{\cancel{8!} \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{12!}{4!\,8!} = \frac{\cancel{12} \times 11 \times \cancel{10} \times 9 \times \cancel{8!}}{\cancel{8!} \times 4 \times 3 \times 2 \times 1}$$

$$= 495$$

The principle of Inclusion - Exclusion (27) :

→ The principle of Inclusion - Exclusion can be stated interms sets as follows

→ let A_1 and A_2 be two sets, let T_1 be the tasks of choosing an element from A_1 and T_2 be the tasks of choosing another element from A_2.

→ there are no. of ways to do task T_1 or task T_2 is

$$\boxed{n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)}$$

where $n(A_1 \cup A_2)$ is the no. of ways to do either tasks T_1 or T_2

$n(A_1 \cap A_2)$ is the no. of ways to do both tasks T_1 and T_2

→ let us take three tasks T_1, T_2 and T_3 let A_1, A_2 and A_3 are three sets. then no. of ways to do this tasks i.e. T_1 or T_2 or T_3 is

$$\boxed{\begin{aligned} n(A_1 \cup A_2 \cup A_3) = {} & n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) \\ & - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3) \end{aligned}}$$

Ex: Consider a set of Integers from 1 to 250 find out

① How many of these numbers are divisible by 3 or 5 from 1 to 250

② " " " " " 5 or 7 1-to-250

③ " " " " " 3 or 7 1-to-250

④ " " " " " 3 or 5 or 7

⑤ " " " " " 3 or 5 but not by 7.

⑥ " " " " " 5 or 7 but not by 3

⑦ " " " " " 3 or 7 but not by 5

⑧ How many of this Integers between 1 and 250 not divisible by 3, 5 and 7.

Sol: let A_1, A_2, A_3 are three sets consisting of Integers that are divisible by 3, 5 and 7 respectively from 1 to 250.

$$n(A_1) = \frac{250}{3} = 83 \rightarrow \text{(taking only Integer part)}$$

$$n(A_2) = \frac{\overset{50}{\cancel{250}}}{\cancel{5}} = 50$$

$$n(A_3) = \frac{250}{7} = 35$$

$$n(A_1 \cap A_2) = \frac{250}{3 \times 5} = 16$$

$$n(A_2 \cap A_3) = \frac{250}{5 \times 7} = 7$$

$$n(A_1 \cap A_3) = \frac{250}{3 \times 7} = 11$$

$$n(A_1 \cap A_2 \cap A_3) = \frac{250}{3 \times 5 \times 7} = 2$$

$$\boxed{\begin{array}{l} n(A_1) = 83,\ n(A_2) = 50,\ n(A_3) = 35,\ n(A_1 \cap A_2) = 16, \\ n(A_2 \cap A_3) = 7,\ n(A_1 \cap A_3) = 11,\ n(A_1 \cap A_2 \cap A_3) = 2 \end{array}}$$

(1) Ans:

$$\begin{aligned} n(A_1 \cup A_2) &= n(A_1) + n(A_2) - n(A_1 \cap A_2) \\ &= 83 + 50 - 16 \\ &= 133 - 16 \\ &= 117 \end{aligned}$$

(2) Ans:

$$\begin{aligned} n(A_2 \cup A_3) &= n(A_2) + n(A_3) - n(A_2 \cap A_3) \\ &= 50 + 35 - 7 \\ &= 85 - 7 \\ &= 78 \end{aligned}$$

(3) Ans:

$$\begin{aligned} n(A_1 \cup A_3) &= n(A_1) + n(A_3) - n(A_1 \cap A_3) \\ &= 83 + 35 - 11 \\ &= 118 - 11 \\ &= 107 \end{aligned}$$

(4) Ans:

$$\begin{aligned} n(A_1 \cup A_2 \cup A_3) &= n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) \\ &\quad - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3) \\ &= 83 + 50 + 35 - 16 - 7 - 11 + 2 \\ &= 170 - 34 \\ &= 136 \end{aligned}$$

⑤ Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_3)$ ← no. of Integers that are divisible by 3 or 5 but not by 7

$= 136 - 35$

$= 101$

(30)

⑥ $= n(A_1 \cup A_2 \cup A_3) - n(A_1)$

$= 136 - 83$

$= 53$

⑦ $= n(A_1 \cup A_2 \cup A_3) - n(A_2)$

$= 136 - 50$

$= 86$

⑧ no. of Integers that are not divisible by 3 or 5 or 7

$n(\overline{A_1 \cup A_2 \cup A_3}) = \text{Total no. of Integers} - n(A_1 \cup A_2 \cup A_3)$

$= 250 - 136$

$= 114$

Ex: Consider a set of Integers from 0 to 499 Found a

① no. of Integers that are divisible by 3 or 5

② " " " " " " 5 or 11

③ " " " " " " 3 or 11

④ " " " " " " 3 or 5 or 11

⑤ " " " " " " 5 or 11 but not by 3

⑥ no. of Integers are divisible by 5 or 11 but not by 11

⑦ " " " " " 3 or 11 but not by 5

(31)

⑧ " " " " not divisible by 3 or 5 or 11

Sol: let A_1, A_2, A_3 are three sets containing the set of Integers that are not divisible by 3 or 5 and 11 respectively. from 0 to 499

$$n(A_1) = \frac{499}{3} = \frac{500}{3} = 166$$

$$n(A_2) = \frac{499}{5} = \frac{500}{5} = 100$$

$$n(A_3) = \frac{499}{11} = \frac{500}{11} = 45$$

Ans: $n(A_1 \cap A_2) = \frac{500}{3 \times 5} = 33$

$$n(A_2 \cap A_3) = \frac{500}{5 \times 11} = 9$$

$$n(A_1 \cap A_3) = \frac{500}{3 \times 11} = 15$$

$$n(A_1 \cap A_2 \cap A_3) = \frac{500}{3 \times 5 \times 11} = 3$$

$$\boxed{\begin{array}{l} n(A_1) = 166, \; n(A_2) = 100 \;\; n(A_3) = 45 \;\; n(A_1 \cap A_2) = 33 \\ n(A_2 \cap A_3) = 9, \;\; n(A_1 \cap A_3) = 15. \; n(A_1 \cap A_2 \cap A_3) = 3 \end{array}}$$

1. Ans: $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$

$= 166 + 100 - 33$

$= 266 - 33 \Rightarrow 233$

2 Ans: $n(A_2 \cap A_3) = n(A_2) + n(A_3) - n(A_2 \cap A_3)$ 32

$= 100 + 45 - 9$

$= 145 - 9$

$= 136$

3 Ans: $n(A_1 \cap A_3) = n(A_1) + n(A_3) - n(A_1 \cap A_3)$

$= 166 + 45 - 15$

$= 211 - 15$

$= 196$

4 Ans: $n(A_1 \cap A_2 \cap A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$

$= 166 + 100 + 45 - 33 - 9 - 15 + 3$

$= 314 - 57$

$= 257$

5 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_1)$

$= 257 - 166$

$= 91$

6 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_3)$

$= 257 - 45$

$= 212$

7 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_2)$

$= 257 - 100$

$= 157$

8 Ans: $n(\overline{A_1 \cup A_2 \cup A_3})$ = total no. of Integers $- n(A_1 \cup A_2 \cup A_3)$

$= 500 - 257$

$= 243$

Example problems on permutations:

① no. of permutations of n-distinct objects is

$$P(n,r) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

② no. of permutations of r- objects among n-distinct objects is

$$P(r,r) = \frac{n!}{(n-r)!} \quad (or) \quad {}^nP_r = P(r,n)$$

③ no.of permutation of n - objects with duplication

$$= \frac{n!}{n_1! \times n_2! \times n_3! \times \ldots \times n_k!}$$

④ Circular permutation :-

- permutations in circular are called circular permutation

- the total no.of ways of arranging the n - persons in a circle is.

$$\text{circle} = (n-1)!$$

Ex: How many ways are there to sit 10 boys and 10 girls arround a circular table?

Sol: Here 10 boys and 10 girls sit around a circular table is

Total no.of persons $= 10 + 10 = 20$

$$\boxed{\therefore n = 20}$$

the ~~total~~ no.of ways of circular permutations are

$$= (n-1)!$$
$$= (20-1)!$$
$$= 19! \text{ ways.}$$

Ex: How many ways are there 3 persons sit round around table ?

Sol: no. of persons = 3

$\therefore n = 3$

total no. of ways of arranging 3 persons around a round table is $(n-1)!$

$= (3-1)!$

$= 2!$

$= 2.$

Ex: How many different arrangements of letters in the word BOUGHT ?

Sol: total no. of letters in given word BOUGHT is = 6. that are distinct.

$n = 6$

Total no. of arrangements of letters in the word BOUGHT $= P(n,n) = n!$

$= 6! = 6\times5\times4\times3\times2\times1$

$= 720$

Example problem on combinations

Ex: A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting atleast 2 questions from each part.

Sol: Question paper contains two parts

part - A → 4-questions

part - B → 4- questions

A student can answer 5 questions, atleast 2 ~~answers~~ question from each part

Case-1: A student can select 3 questions from part-A, & 2 questions from part-B.

→ 3. Question from part-A can be selected $C(4,3)$ ways

→ 2 questions from part-B can be selected $C(4,2)$ ways

∴ Number of ways = $C(4,3) \times C(4,2)$

$= \frac{4!}{3!(4-3)!} * \frac{4!}{2!(4-2)!}$ (37)

$= \frac{4\times\cancel{3!}}{\cancel{3!}\times1!} \times \frac{\cancel{4}^{2}\times3\times\cancel{2}\times1}{\cancel{2!}\times\cancel{2}} \Rightarrow 4*6 = 24$ ways.

case-II: A student can select 2 questions from part-A & 3 questions from part-B.

2-questions are selected from part-A in $C(4,2)$

3-question are selected from part-B in $C(4,3)$

$\therefore$ no. of ways $= C(4,2) * C(4,3)$

$= \frac{4!}{(4-2)!\,2!} * \frac{4!}{3!(4-3)!}$

$= \frac{\cancel{4}^{2}\times3\times\cancel{2}\times1}{\cancel{2!}\cancel{2!}} * \frac{4\times\cancel{3!}}{\cancel{3!}\times1!} = 6\times4$

$= 24$

total no. of ways, A student can answer 5 questions by selecting atleast 2 questions from each part $= 24 + 24$

$= 48$ ways.

ex: How many committees of 5 with a given chair person can be selected from 12 persons

Sol: Total no. of persons = 12 (38)

each committee consisting of 5 persons

Among them, 1 person is chair person.

the chair person can be selected among 12 persons = 12 ways.

the remaining 4 persons in the committee can be selected in $11C_4$ ways.

∴ total no. of ways $= 12 * 11(C,4)$

$= 12 * \frac{11!}{4! * (11-4)!}$

$= 12 * \frac{11!}{4! * 7!} = 12 * \frac{11 \times 10 \times 9 \times 8 \times 7!}{4! \times 7!}$

$= 12 * 330$

$= 3960$ ways.

22: Find the no. of committees of 5 that can be selected from 7 men and 5 women if the committe is to consists of atleast one men and atleast one women

(39)

Sol: Total no. of persons = 7 - men + 5 - womens
= 12 persons

each committee consists of 5 persons.

No. of committees of 5 can be selected among 12 persons = $c(12, 5)$

Among this possible committes. no. of ways committes to select 5 men = $c(7,5)$ and no. of committes formed to select 5 woment is = $c(5,5)$

∴ total no. of committes formed with atleast one man and one women =

$\frac{12!}{5!(12-5)!}$

$= c(12,5) - c(7,5) - c(5,5)$

$\frac{12!}{(12-5)!5!} - \frac{7!}{5!(7-5)!} - \frac{5!}{5!(5-5)!}$

$= 792 - 21 - 1$

$= 770$

Ex: Find a group of 7 men and 6 women five persons are to be selected to form a committee so that atleast 3 men are on the committe In how many ways it can be done?

40

Sol: Each committee consists of 5 persons

Total no. of men = 7

" " women = 6

no. of ways to form a committee with atleast 3 men are = ① 3 men & 2 women

② 4 men & 1 women

③ 5 men & 0 women

no. of ways to form a committee with atleast 3 men are $= (7c_3 * 6c_2) + (7c_4 * 6c_1) + (7c_5 * 6c_0)$

$= \frac{7!}{(7-3)!\,4!} * \frac{6!}{(6-2)!\,2!} + (7c_4 \times 6c_1) + (7c_5 \times 6c_0)$

$= (35 \times 15) + (35 \times 6) + (21 \times 1)$

$= 525 + 210 + 21$

$= 756$ ways.

Ex: How many different strings of length 4 can be formed using the letters of the word PROBLEM? (41)

Sol: the given word "problem" has 7 letters

$n = 7.$

the no. of different string of length 4 can be formed by sing the letters of the word PROBLEM $= P(n, r)$

$P(n,r) = P(7,4) = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}}$

$= 840$ ways.

UNIT-V
Graph Theory

Graph:

A graph G is a pair of sets (V, E) where V is a set of vertices and E is a set of edges.

The most common representation of a graph is a diagram with vertices and edges.

(i) The vertices (or) nodes are represented as points or small circles.

(ii) Edges are represented as line segments (or) curve joining of its end vertices.

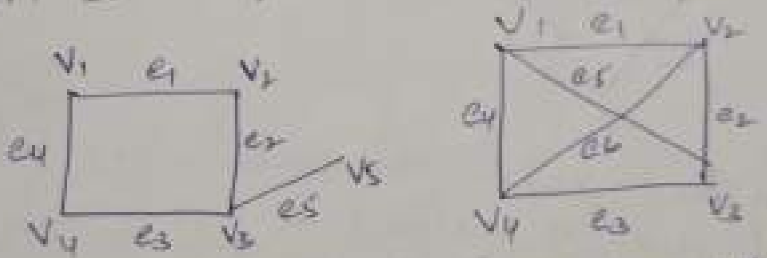


fig: Representation of a graph with vertices and edges

① The first graph consists of 5 vertices and 5 edges

so

$V = \{V_1, V_2, V_3, V_4, V_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$

② In the second graph, there are 4 vertices and 6 edges. so,

$V = \{V_1, V_2, V_3, V_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

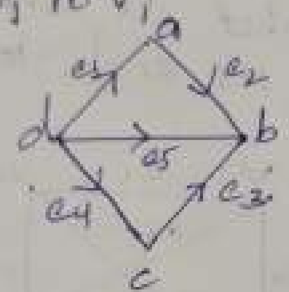
There are 2 types of graphs They are
(1) Directed graph
(2) undirected graph

Directed graph :

The graph in which the elements of the edges set are ordered pairs of vertices is called directed graph or digraph.

Here order pair (V_i, V_j) denotes an edge from vertex, V_i to vertex V_j (V_j, V_i) denotes an edge from V_j to V_i

Eg :



directed graph

Here in this graph, elements edge set are ordered pair of vertices is

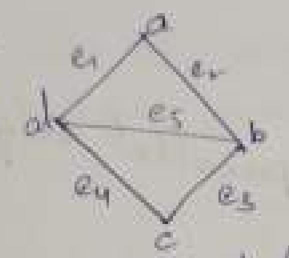
$e_1 = (d, a)$ $e_4 = (d, c)$
$e_2 = (a, b)$ $e_5 = (d, b)$
$e_3 = (c, b)$

undirected graph .

A graph in which the elements of the Edges set are unordered pair of vertices is called undirected graph (non-directed)

Here (V_i, V_j) denotes an edge from between V_i, V_j

Eg:-



undirected graph

Then $e_1 = \{a,d\}$ $e_5 = \{b d\}$
$e_2 = \{a,b\}$
$e_3 = \{b,c\}$
$e_4 = \{c,d\}$

<u>Null graph</u> :

A graph in which no. of edges is zero is called as Null graph

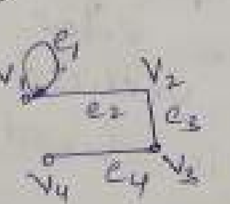
Eg

Null graph with 4 Vertices and zero edges

<u>Self loop</u> :

An edge joining a vertex to itself is called as selfloop

Eg:

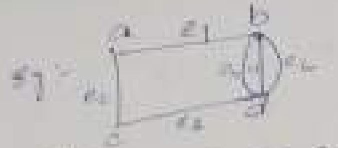


A graph with a selfloop. In this graph edge e_1 is a selfloop

<u>Parallel & multiple edges</u> :

In a graph it may be possible to have more than one edge with a single pair of Vertices ; such edges are called parallel edges

Eg:-



In Example e_4, e_5, e_6 are parallel edges

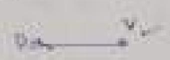
<u>Simple graph</u>

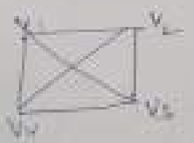
A graph which contains neither selfloop nor parallel edges is called a simple graph

Eg:

<u>complete graph</u>

A simple graph in which there is exactly one edge between each pair of distinct vertices is called a complete graph. It is denoted by k_n





- The no of edges in the complete graph with n vertices $= \frac{n(n-1)}{2}$

Eg:- Find the total no. of edges of a complete graph with 50 vertices.

The total no of edges of a complete graph with 50 vertices $= \frac{50(50-1)}{2} = 1225$

<u>Multigraph</u>:

A graph which contain parallel edges is called multigraph.

Eg.

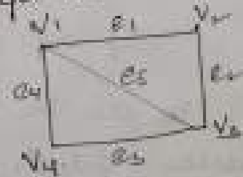
<u>Order and size of a graph</u>

The number of vertices in a graph G is called order of the graph

It is denoted by $|V(G)|$

The no. of edges in a graph G is called size of the graph and It is denoted by $|E(G)|$

Eg:-



order of the graph is $|V(G)| = 4$

Size of the graph is $|E(G)| = 5$

* Degree of vertex in a Non-directed graph and degree sequence

The degree of a vertex v in a graph G is denoted by $deg_G(v)$

The degree of a vertex v of a graph G is the no of edges of G which are incident with v

Isolated vertex:

A vertex of degree zero is called a isolated vertex.

Pendant vertex:

A vertex with degree one is called pendant vertex.

odd vertex:

A vertex with odd degree is called odd vertex.

Even vertex

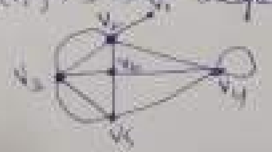
A vertex of even degree is an even vertex

Note

The degree of self loops is counted twice

If $v_1, v_2, v_3, \ldots v_n$ are the vertices of G, then the sequence $\{d_1, d_2, \ldots d_n\}$ where $d_i = deg_G(v_i)$ is the degree sequence of G

eg



degree of vertex $v_1 = 1$
degree of vertex $v_2 = 5$
degree of vertex $v_3 = 5$
degree of vertex $v_4 = 5$
degree of vertex $v_5 = 4$
degree of vertex $v_6 = 4$

Degree sequence of graph is given by $\{1, 5, 5, 5, 4, 4\}$

* Degree of vertex in a directed graph

Indegree

The no of edges incident to a vertex is called the Indegree of the vertex for a graph

outdegree

The no of edges incident from it is called the outdegree for a digraph

The indegree of a vertex v in a graph G is denoted by $deg_G^+(v)$

The outdegree of vertex v in a graph G is denoted by $deg_G^-(v)$

The degree of a vertex is determined by counting each loop incident on it twice and each other edge once

The minimum of all the degrees of the vertices of a graph G is denoted by $\delta(G)$

The maximum of all the degrees of the vertices of a graph G is denoted by $\Delta(G)$

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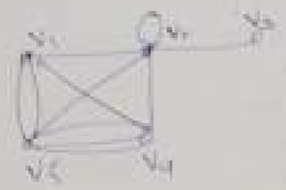
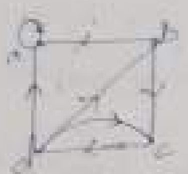


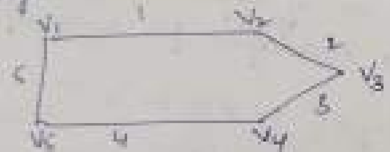
$deg(v_1) = 4$ $deg(v_4) = 5$
$deg(v_2) = 6$ $deg(v_5) = 6$
$deg\ v_3 = 1$

(ii) 

$Deg^+(a) = 3$ $Deg^-(a) = 1$
$Deg^+(b) = 1$ $Deg^-(b) = 2$
$Deg^+(c) = 2$ $Deg^-(c) = 1$
$Deg^+(d) = 1$ $Deg^-(d) = 3$

Weighted graph :

A graph in which weights are assigned to every edge is called a weighted graph.



Here 1,2,3,4,5 are weights assigned to each edge respectively.

Path :

In a non directed graph G, a sequence 'P' of zero or more edges of the form $\{v_0, v_1\}, \{v_1, v_2\}, \ldots \{v_{n-1}, v_n\}$ & $v_0, v_1, v_2, v_3 \ldots v_n$ is called Path from v_0 to v_n where

v_0 — Initial vertex
v_n — terminal vertex of the path P

(1) In a path, vertices and edges ~~may~~ not to be repeated any no of times.

(2) The no of edges in a path is called length of the path.

Trivial path

A path of length zero is called trivial path.

Path	Length
a-b-c-d	3
a-b-c-d-e	4
a-b-c-d-e-a	5
a-b-c-d-e-a	4
a-b-c-d-a	3
a-b-d-e	3
b-c-d-b	0
a	1
a-b	

Open path

A path in which initial and terminal vertices are distinct is called open path

Eg:- a-b-c-d is open path

Closed path :

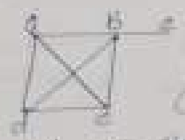
A path in which the initial and terminal vertices are same is called closed path

Eg:- a, a-b-c-d-a, a-b-d-e-a

Simple path

A path is said to be simple if all the edges and vertices on path are different except possibly of the end point

Eg:



← simple path representation
Here the different paths are
(1) a-b-c-a (3) a-b-c-d-a
(2) a-b-e-c-d-c-a (4) a-b-c-a-b

Here (1)-(3) are simple paths, where as (2) and (4) not simple path

Cycle graph

A cycle graph of order n is a connected graph whose edges forms a cycle of length 'n'

It is denoted by 'C_n'

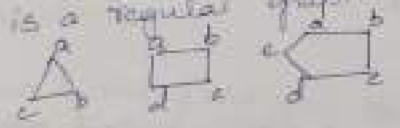
Cycle graph of order 'n' vertices with

(1) In a cycle graph have n vertices and n edges. starting & ending vertices are same

(2) In C_n, $\deg(v_i) = n-1$ $\forall$ one and every

C_n is a regular graph

eg



Wheel graph

A wheel graph of order 'n' is a graph obtained by joining a single new vertex to each vertex of cycle graph (C_{n-1}) of order $(n-1)$

It is denoted by W_n

The no. of vertices in a wheel graph is $n+1$ and edges $2n$

In W_n, $|V| = n$
$|E| = 2(n-1)$

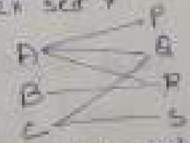
E.g

** Bipartite graph

A simple graph G is such that its vertex set V is the union of two mutually disjoint non-empty sets V_1 and V_2 which are such that every edge in G joins a vertex in V_1 and a vertex in V_2. Then G is called a bipartite graph

If E is the edge set of this graph, the graph is denoted by $G = (V_1, V_2, E)$ & $G = G(V_1, V_2, E)$

The sets V_1 and V_2 are called bipartites (a partition) of the vertex set V



In the above graph G, vertex set $V = \{A, B, C, P, Q, R, S\}$ V is the union of V_1 and V_2

Edge set $E = \{AP, AQ, AR, BR, CQ, CS\}$

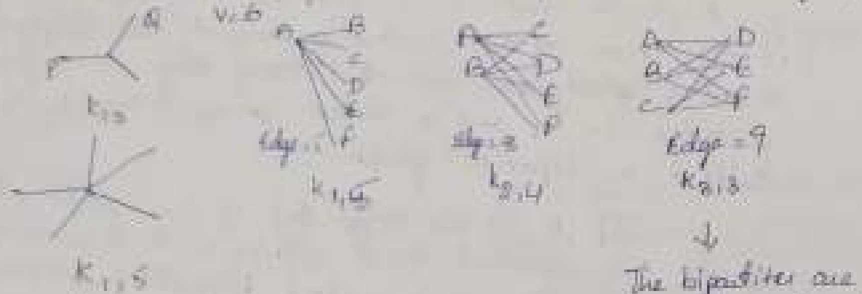
V_1 and V_2 are bipartites

Complete bipartite graph

A complete bipartite graph $G = (V_1, V_2, E)$ in which the bipartites V_1 and V_2 contains m and n vertices respectively with $m \le n$ is denoted by $K_{m,n}$

A bipartite graph $G = (V_1, V_2, E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2

In this graph each of m vertices in V_1 is joined to each of n vertices of V_2. Thus $K_{m,n}$ has $m+n$ vertices and mn edges i.e., $K_{m,n}$ is of order $m+n$ and size mn. It is therefore a $(m+n, mn)$ graph



$K_{1,3} = V_1 \{a\}$ $V_2 = \{P, Q, R\}$

The bipartites are

$V_1 = \{A, B, C\}$ & $V_2 = \{P, Q, R\}$

vertices   vertices

The graph $K_{3,3}$ is of great important. This is known the kuratowski's second graph

Representation of graph

A matrix is a convenient and useful away of representing a graph. many known result of matrix algebra can be applied to study the properties of graphs and to calculate paths, cycles and other

Characteristics of a graph

3 type of representing

1) Adjacency matrix
2) Incidence matrix
3) Pathmatrix of a graph

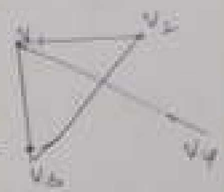
1) Adjacent matrix

$G(V, E)$ be a simple graph with n vertices ordered from V_1 to V_n then the adjacency matrix

$A = [a_{ij}]_{n \times n}$

$n \times n$ symmetric matrix defined by the elements

$$a_{ij} = \begin{cases} 1 & V_i \text{ is adjacent to } V_j \\ 0 & \text{otherwise} \end{cases}$$



$A(G)$ & A_G

$(V_i, V_j) \in E$ G adjacent matrix of directed graph G graph

$$\begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 1 & 1 \\ V_2 & 1 & 0 & 1 & 0 \\ V_3 & 1 & 1 & 0 & 0 \\ V_4 & 1 & 0 & 0 & 0 \end{array}$$

Adjacency matrix in case of directed graph

$$\begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 0 & 0 \\ V_2 & 0 & 0 & 1 & 0 \\ V_3 & 0 & 0 & 0 & 1 \\ V_4 & 1 & 0 & 0 & 0 \end{array}$$

Problems

1) Draw a graph with the given adjacent matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Sol:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 1 & 1 \\ b & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 1 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

8) without drawing the graph, Prove that the graph where adjacency matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$ is connected

Sol: adjacent matrix is

$$X = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Order = 5 = vertices

$$X^2 = X \cdot X = \begin{bmatrix} 2 & 1 & 0 & 1 & 2 \\ 1 & 3 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 1 \\ 2 & 1 & 0 & 1 & 3 \end{bmatrix}, \quad X^4 = X^3 \cdot X = \begin{bmatrix} 10 & 9 & 3 & 9 & 11 \\ 9 & 16 & 6 & 15 & 10 \\ 8 & 6 & 3 & 6 & 2 \\ 9 & 10 & 6 & 16 & 9 \\ 12 & 10 & 2 & 10 & 11 \end{bmatrix}$$

$$X^3 = X^2 \cdot X = \begin{bmatrix} 2 & 5 & 2 & 5 & 2 \\ 6 & 4 & 1 & 6 & 6 \\ 2 & 1 & 0 & 1 & 2 \\ 5 & 5 & 1 & 4 & 2 \\ 2 & 6 & 3 & 2 & 2 \end{bmatrix}$$

Every entry in X^4 is non-zero

∴ no entry in $Y = X + X^2 + X^3 + X^4$ can be zero

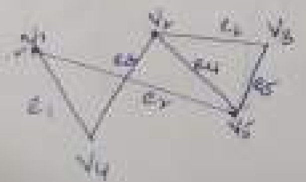
Hence given graph is connected

(2) Incidence matrix

undirected graph

Let G be a graph with n vertices $V = \{v_1, v_2, \ldots v_n\}$ and $E = \{e_1, e_2, \ldots e_n\}$. Define $n \times m$ matrix $I_G = [m_{ij}]_{n \times m}$ where $m_{ij} = \begin{cases} 1 & \text{which } v_i \text{ has incident } e_j \\ 0 & \text{otherwise} \end{cases}$

Eg:-



$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 1 & 0 \end{array} = I_G$$

Directed graph:

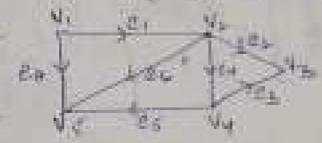
$V = \{v_1, v_2, \ldots v_n\}$

$E = \{e_1, e_2, \ldots e_n\}$

$B = b_{ij}$ defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of the edge } e_j \\ -1 & \text{if } v_i \text{ is final vertex of } e_j \\ 0 & \text{otherwise} \end{cases}$$

Eg:-



$$B(G) = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ v_2 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{array}$$

3) Path matrix / reachability matrix

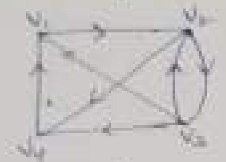
Let G be a simple digraph having no parallel directed edges and $V = \{v_1, v_2, \ldots v_n\}$ be its vertex set. An $m \times n$ matrix $P = [P_{ij}]_{n \times n}$ is given by

Let $G = (V, E)$ be a simple digraph in which $|V| = n$ and nodes of G are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

$$P_{ij} = \begin{cases} 1 & \text{if there is a path from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

is called a pathmatrix of G

Eg :- consider the graph find path matrix



Adjacent matrix

$$A = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline V_1 & 0 & 1 & 0 & 0 \\ V_2 & 0 & 0 & 1 & 1 \\ V_3 & 1 & 0 & 1 & 1 \\ V_4 & 1 & 0 & 0 & 0 \end{array}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad A^4 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$B_n = A + A^2 + A^3 \ldots$

$B = A + A^2 + A^3 + A^4$

$$B = \begin{bmatrix} 5 & 4 & 2 & 3 \\ 5 & 3 & 4 & 6 \\ 4 & 4 & 4 & 7 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

All the vertices in the graph are connect with different paths

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

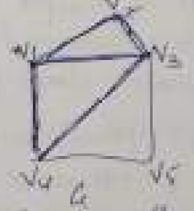
Subgraphs :

Given two graphs G and G_1, we say that G_1 is a Subgraph of G. If it follows conditions that:

(1) All the vertices and all the edges of G_1 are in G

(2) Each edge of G_1 has the same end vertices in G as in G_1

Eg



all vertices and edges of the graph G_1 are in graph G and that Every edge in G_1 has the same End vertex in G as in G_1

$\therefore$ G_1 is a subgraph of G

Note :-

(a) Every graph is a subgraph of itself

(b) Every simple graph of n vertices is a subgraph of the complete graph K_n

(c) G_1 is a subgraph of graph G_2 and G_2 is a subgraph of a graph G_1, then G_1 is a subgraph of G

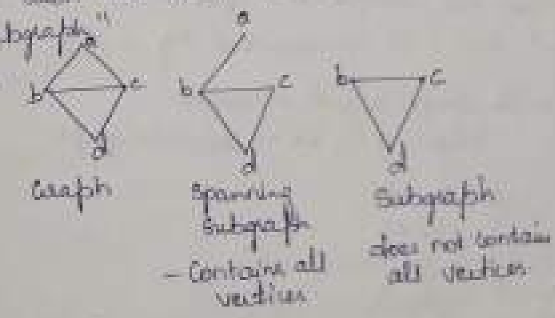
(d) A single vertex in a graph G is a subgraph of G

(e) A single edge in a graph G together with its end vertices is a subgraph of G.

Spanning subgraph

Graph $G = (V, E)$ if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$ then G_1 is called the "Spanning Subgraph"

Eg



Induced Subgraph

If a graph $G = (V, E)$ and there subgraph $G_1 = (V_1, E_1)$ of G such that every edge {A, B} of G where $A, B \in V_1$ is an edge {A, B} of G_1 also

Then G_1 is called a subgraph of G induced by V_1 and denoted by $\langle V_1 \rangle$

graph (a) and (b) is an induced subgraph of set of vertices $\{V_1, V_2, V_3, V_5\}$

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* Edge disjoint and vertex disjoint subgraphs
(1) G_1 and G_2 graph are said to be edge disjoint if they dont have any common edge
(2) vertex disjoint – donot have any common edge and any common vertex

Complement of a graph

Every simple graph of order n is a subgraph of the complete graph k_n. If G is a simple graph of order n, Then the complement of G in k_n is called the complement of G. It is denoted by G' & $\bar{G}$

Simple graph G with n vertices
edges of $\bar{G}$, $k_n \rightarrow$ complete graph

$\bar{G} = k_n - G$
$\bar{G} = k_n \Delta G$

Eg:

(a) Kn (b) G (c) $\bar{G}$

Complement of a subgraph

Given a graph G and a subgraph G_1 ~~and of G~~, the subgraph of G obtained by deleting from G all the edges that belong to G_1 is called "complement of G_1 in G"
$G - G_1$ & $\bar{G}_1$

Eg:- $\bar{G}_1 = G - G_1$

(a) G_n (b) G' (c) $\bar{G}$

Isomorphism

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A function $f: G_1 \rightarrow G_2$ is called an isomorphism. If
(1) If f is one to one and onto
(2) If the graph G_1 is isomorphic to G_2. Then we write $G_1 \cong G_2$

Properties

If 2 graphs G_1 and G_2 are isomorphic then
(a) $|V(G_1)| = |V(G_2)|$
(b) $|E(G_1)| = |E(G_2)|$
(c) $\deg G_1(V) = \deg G_2(V)$
degree of sequence of G_1 and G_2 are the same
(d) If 2 graphs are isomorphic then their adjacency matrices are same

Eg: Let us consider the graphs

G G_1

(1) no of vertices of G and G_1 & edges & degree(v) of G and G' are same

	G	G'
a)	$\lvert V\rvert = 4$	$\lvert V\rvert = 4$
b)	$\lvert E\rvert = 6$	$\lvert E\rvert = 6$
c)	deg(A) = 3	deg(P) = 3
	deg(B) = 3	deg(Q) = 3
	deg(C) = 3	deg(R) = 3
	deg(D) = 3	deg(S) = 3

the following one to one mapping of vertices and adjacent vertices degree must be same

$A \leftrightarrow P$
$B \leftrightarrow Q$
$C \leftrightarrow R$
$D \leftrightarrow S$

mapping of edges

$\{A,B\} \leftrightarrow \{P,Q\}$ $\quad \{B,C\} \leftrightarrow \{Q,R\}$
$\{A,C\} \leftrightarrow \{P,R\}$ $\quad \{B,D\} \leftrightarrow \{Q,S\}$
$\{A,D\} \leftrightarrow \{P,S\}$ $\quad \{C,D\} \leftrightarrow \{R,S\}$

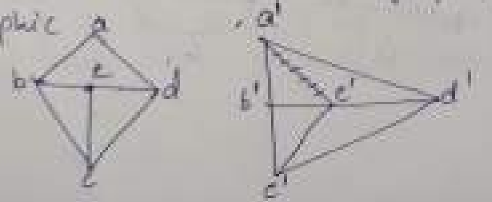
adjacent matrices are equal

$$G = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad G_1 = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

∴ The given 2 graphs are Isomorphic

Problems

(1) Determine whether the following graphs are Isomorphic



Sol: no of vertices, edges and degree(v) must be same

	G	G'
1)	$\lvert V\rvert = 5$	$\lvert V\rvert = 5$
2)	$\lvert E\rvert = 7$	$\lvert V\rvert = 7$
3)	deg(a) = 2	deg(a') = 2
	deg(b) = 3	deg(b') = 3
	deg(c) = 3	deg(c') = 3
	deg(d) = 3	deg(d') = 3
	deg(e) = 3	deg(e') = 3

All vertices
edges
degree of each vertices of
G and G' are same

mapping of vertices of adjacent vertices degree

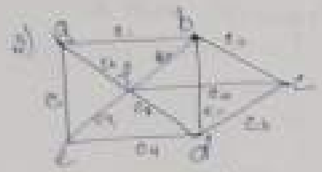
$a \leftrightarrow a'$
$b \leftrightarrow b'$
$c \leftrightarrow c'$
$d \leftrightarrow d'$
$e \leftrightarrow e'$

mapping of ~~vertices~~ edges.

$\{a,b\} \leftrightarrow \{a',b'\}$ $\quad \{c,e\} \leftrightarrow \{c',e'\}$
$\{a,d\} \leftrightarrow \{a',d'\}$ $\quad \{c,b\} \leftrightarrow \{c',b'\}$
$\{d,c\} \leftrightarrow \{d',c'\}$ $\quad \{b,e\} \leftrightarrow \{b',e'\}$

$$G_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad G_2 = \begin{array}{c} \\ a' \\ b' \\ c' \\ d' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Scanned with ACE Scanner

3) 

graph (G)

graph (G')

G	G'
$\lvert V \rvert = 6$	$\lvert V \rvert = 6$
$\lvert E \rvert = 11$	$\lvert E \rvert = 11$
deg(a) = 3	deg(a') = 4
deg(b) = 4	deg(b') = 3
deg(c) = 3	deg(c') = 4
deg(d) = 4	deg(d') = 3
deg(e) = 3	deg(e') = 3
deg(f) = 5	deg(f') = 5

vertices, edges and degree of vertices are same.

mapping of vertices.

a ⟷ c'
b ⟷ a'
c ⟷ b'
d ⟷ e'
e ⟷ d'
f ⟷ f'

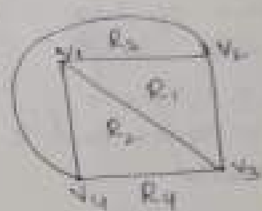
mapping of edges.

ab ⟷ c'a'
ae ⟷ c'd'
af ⟷ c'f'
bc ⟷ a'b'
bd ⟷ a'e'
bf ⟷ a'f'
cd ⟷ b'e'
cf ⟷ b'f'
de ⟷ e'd'
df ⟷ e'f'
ef ⟷ d'f'

Planar graph

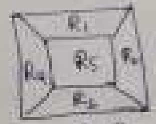
A graph is called planar if it can be drawn in the plane without any edges crossing is called planar graph. And a graph that cannot be drawn on a plane without a crossover b/w its edges is called Non-planar graph.

Non-planar



not crosses & intersect

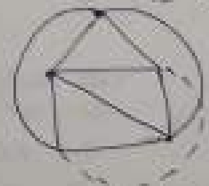
$\underbrace{R_1, R_2, R_3}_{\text{connected graph}}, \underbrace{R_4}_{\text{disconnected graph}}$ → Regions

(ii) 

R_6 Planar graph

$\underbrace{R_1, R_2, R_3, R_4, R_5}_{\text{connected graph}}, \underbrace{R_6}_{\text{disconnected graph}}$

Theorem: A complete graph of five vertices is Non planar.



→ Non-planar for planar there must not be any intersection & crossing.

walk and their classification:

Five important subgraph are called a walk, a path, a trail, a circuit and a cycle.

Walk

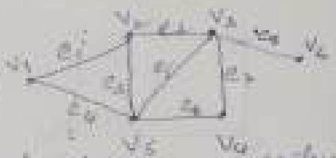
In a graph G have atleast one edge
- alternate sequence of vertices and edges

$v_i e_j v_{i+1} e_{j+1} v_{i+2} \ldots e_k v_m$

which starts with vertices and ends with vertex
- incidence on edge on the vertices no of edges present
in walk = length

$v_1 e_1 v_2 e_2 v_3 e_8 v_6$

no vertex and no edges repeated



A walk that begins and ends at the same vertex is called a closed walk which is not closed is called an open walk

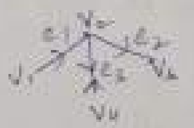
Trail and circuit

In walk - vertices ~~and (& edges)~~ may appear more than once

If an open walk no edge appears more than once then the walk is called a trail

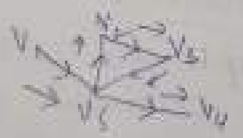
A closed walk in which no edge appear more than once is called a circuit

eg.



$v_1 - v_2 - v_3$

All vertices and edge are not covered
One edge left
Open walk



Circuit

Closed walk
no edge appear more than once

Not a circuit
Both direction on one edge

$v_1 - v_2 - v_3 - v_4 - v_2 - v_4 - v_1$

edges and vertices are repeated

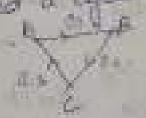
$v_1 - v_2 - v_5 - v_3 - v_4 - v_5 - v_1$

Circuit

Path and cycle

A trail in which no vertex appears more than once is called a path
- Open walk
no edge, no vertex appears more than once
- Path

A circuit in which the terminal vertex does not appear as an internal vertex and no internal vertex is repeated is called a cycle



A - B - C - A
cycle

A-B - C - D - E - C - A
C is repeated
not a cycle

Facts

walk - open/closed walk
- vertex & edge can appear more than once

trail - open walk
- vertex & edge can appear more than once
edge cannot appear more than once

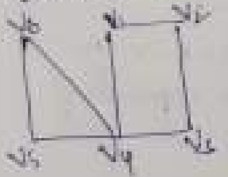
Circuit - closed walk
vertex can appear more than once
but an edge cannot appear more
than once.

Path — open walk
neither vertex nor edge can appear
more than once.
Path = trail
trail ≠ path.

Cycle — closed walk
- neither a vertex nor an edge can be
appear more than once.
cycle = circuit circuit ≠ cycle

- cycle — which is only one edge - loop
- ll'el edges form a cycle.
- simple graph, a cycle must have atleast
three edges.

Ex:- Indicate the nature of the following walks.



(i) $v_1 v_2 v_3 v_4$ open walk which is not a trail.

(ii) $v_1 v_2 v_3 v_4 v_5$
no edges repeated & vertices
trail which is a path

(iii) $v_6 v_5 v_4 v_3 v_2 v_1 v_4 v_6$

closed walk
- Circuit but not a cycle.

Planar graph

Connected and disconnected graphs
→ A graph G of order greater than or equal to two
order ≥ 2
→ vertices

Two vertices in G are said to be connected
if there is atleast one path from vertex to
other.
connected - every pair of vertex is connected
if it is not connected = disconnected.

Theorem 2: A simple graph with n vertices and
k components can have at most $(n-k)(n-k+1)/2$
edges.

Proof:- $n_1 \rightarrow$ no of vertices in the first component
$n_2 \rightarrow$ no. of " " " second "
⋮
n_k " " " " " k^{th} "

$n_1 + n_2 + n_3 + \cdots n_k = n \rightarrow (1)$
$(n_1-1)+(n_2-1)+(n_3-1)+(n_4-1)+\cdots(n_k-1)$
$= n-(1+1+1+1+\cdots 1)$
$= n-k$

Squaring both sides
$(n_1-1)^2+(n_2-1)^2+(n_3-1)^2+(n_4-1)^2+\cdots(n_k-1)^2 = n+$
∴ 3 - sum of the product of the three

$2(n_i-1)(n_j-1)$ $\quad i = 1,2,3,\ldots k \quad i \neq j$

$j = 1,2,3,\ldots k$

each of $n_1, n_2, \ldots n_k \geq 1 \quad \leq 0$

$(n_1-1)^2 + (n_2-1)^2 + \ldots + (n_k-1)^2 \leq (n-k)^2$

$\xrightarrow{} 2$

or $n_1^2 + n_2^2 + n_3^2 + \ldots n_k^2 - 2(n_1 + n_2 + n_3 + \ldots n_k)$

$+ k \leq (n-k)^2$

$n_1^2 + n_2^2 + \ldots n_k^2 \leq (n-k)^2 + 2n - k$

$\leq n^2 + k^2 - 2nk + 2n - k$

$\leq n^2 - k(k-2n) - 1(k-2n)$

$= n^2 - (k-1)(k-2n)$

$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$

G is a simple graph, each of the components of G is a simple graph.

Max no. of edges which is i^{th} component can have is $\frac{1}{2} n_i(n_i-1)$

Give in $N = \frac{1}{2} \sum_{i=1}^{k} n_i(n_i-1)$

$\xrightarrow{} 4$

$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i$

$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n$ using 1 using

$\leq \frac{1}{2} (n^2 - (k-1)(2n-k)) - \frac{1}{2} n$

$\xrightarrow{} 3$ using

$= \frac{1}{2} (n^2 - 2nk + n + k^2 - k)$

$= \frac{1}{2} (n-k)(n-k+1)$

no of edges cannot exceeds $\frac{1}{2}(n-k)(n-k+1)$

Euler's circuit and Euler trails

- Connect graph G
- there is a circuit in G contains all the edges then circuit - Euler circuit (Eulerian line / Euler tour)
- If there is trail in G that contains all the edges of G - that trail - Euler trail

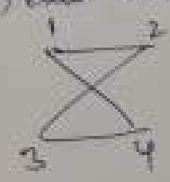
Circuit — no edges can appear more than once vertex can appear more than once vertex can appear more than once

- Euler circuit and Euler trail - include all the edges
- A connected graph that contains an Euler circuit is called an Euler graph
- Euler path: It is a path that traversal each edge exactly once and only once
- A graph that contains an Euler path is called as Euler graph.
- Euler graph is always connected because Euler path contains all the edges of the graph
- Euler circuit: First and last vertex are same. It is a circuit that traverses each edges exactly only once.

** → A connected graph is a Euler graph if it has atmost 2 odd degree vertices.

→ Euler path → max two vertices odd degree.

→ Euler circuit → each vertices is of even degree



1 - 2 - 3 - 4 → Euler path

1 - 2 - 3 - 4 - 1 → Euler circuit

→ all vertices connected Euler graph.

→ Euler's

3) 

a-b-c-d

Euler's path no circuit

deg(a) = 1 deg(b) = 3 } Euler's path.
deg(c) = 2 deg(d) = 2

8) 

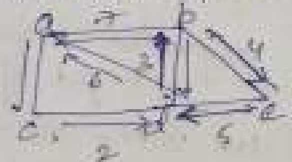
Start at a end at c

open Euler walk - a Euler walk which visit every edge of the graph exactly once.

semi Euler walk if a graph contains a open Euler walk

Solved Numericals on euler graph

(1) Find Euler's path for graph below



a-c-d-b-e-d-a-b

Euler's path.

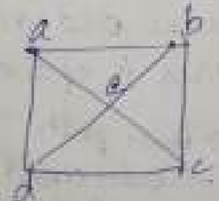
a-3, b-3, c-2, d-4, e-2

two odd degree.

(2) 

$V_1 - V_2 - V_3 - V_5 - V_4 - V_3 - V_1$

all have Euler's circuit then even degree Euler graph

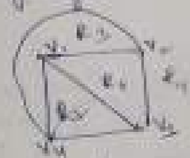
(3) 

edge is going to repeat

Euler's formula:

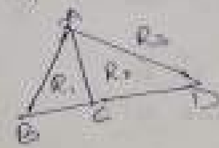
Let G be a connected planar simple graph with e edges and vertices 'v'. Let 'r' be the no. of regions in planar rep^n of G then

$\boxed{r = e - v + 2}$



v = 4
e = 6

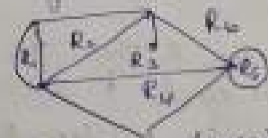
r = 6 - 4 + 2
r = 2 + 2
$\boxed{r = 4}$



v = 4
e = 5
r = 5 - 4 + 2
= 1 + 2
$\boxed{r = 3}$

Diagram divides the plane into a no. of parts called regions (faces) of which exactly one part is unbounded.

→ The no. of edges that form the boundary of a region is called of degree of that region



$\underbrace{R_1, R_2, R_3, R_4, R_5}_{\text{bounded}}$, R_6 → divided
R_6 → unbounded / exterior

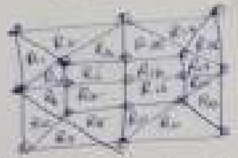
deg(R_1) = 2, deg(R_2) = 3, deg(R_5) = 1, deg(R_3) = 4
deg(R_4) = 3, deg(R_6) = 5

2 + 3 + 4 + 3 + 5 = 18

generally here we follow the handshaking property. it true for all planar graph

2 + 3 + 2 + 4 + 3 + 6 = 20

$V = 6 = n$ $\quad n - m + r = 6 - 10 + 6$
edges $= 10$ $\quad = -4 + 6 = 2$
$r = 6$



R19 $\quad n = 17 \quad r = 19 \quad n - m + r = 17 - 34 + 19$
$m = 34$ $\quad = -17 + 19 = 2$

If there is pendent vertex it counts twice

* Let G be a 4-regular connected planar graph having 16 edges. Find the no. of region of G.
* Let n be the no of vertices in G each vertices having degree as 4

sum of degree $= 4n$

$\sum_{i=1} deg(v_i) = 2|E|$
$= 32$

$\therefore E = 16$
$4n = 32$
$n = 8$

$r = m - n + 2$
$= 16 - 8 + 2$
$= 8 + 2 = 10$

Hamilton cycles and Hamilton paths

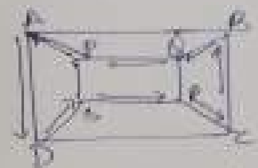
- G is a connected graph
- Suppose there is a cycle in G then the cycle is called a Hamilton cycle in G.
- Famous Irish mathematician Sir. William Hamilton
- A Hamilton cycle in a graph of n vertices consists of exactly n edges. Because a cycle with n vertices has n edges.

⇒ It means it includes all vertices in G
it not necessary to include all edges of G

A graph contains a Hamilton cycle is called a Hamilton graph

Ex:-

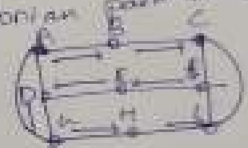
A-B-C-D-A
Hamilton cycle
↳ Hamilton graph



A-D-S-R-C-B-Q-P-A
Hamilton cycle → Hamilton graph

A path (if any) in a connected graph which includes every vertex (but not necessarily every edge) of graph is called a Hamilton path in a graph

- Hamiltonian path



A B C D E F G H I
↓
Hamilton path
not Hamilton cycle

The length of Hamilton path in a connected graph of n vertices is n-1 edges.

Graph coloring:

Coloring a graph constitutes coloring vertices/ edges of the graph.

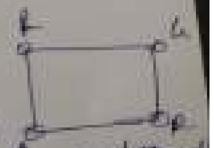
coloring all the vertices of a graph in the property that no two adjacent vertices have same color.

Chromatic Numbers

It is defined as the least no. of colors needed for coloring the graph - denoted by $\chi(G)$

K-chromatic graph

3 color

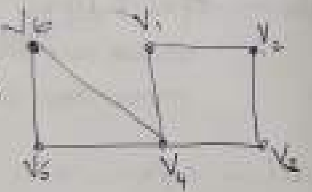


2-chromatic graph

Cycle — closed walk
- neither a vertex nor an edge can appear more than once
cycle = circuit + cycle.

- cycle — only one edge — loop
- n edges form a cycle
- simple graph, a cycle must have atleast three edges

Ex: Indicate the nature of the following walks



(i) $V_1 V_2 V_3 V_2$
open walk which is not a trail

(ii) $V_1 V_2 V_3 V_4 V_5$

no edges repeated & vertices
— trail which is a path

(iii) $V_6 V_5 V_4 V_3 V_2 V_1 V_4 V_6$

closed walk
— circuit but not a cycle.

Connected and Disconnected graphs

⇒ A graph G of order greater than & equal to two

order ≥ 2
↳ vertices

Two vertices in G are said to be connected if there is at least one path from one vertex to the other.

Connected — every pair of vertex is connected
If it is not connected = disconnected

Theorem: A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges

Proof $n_1 \rightarrow$ no of vertices in the first component.
$n_2 \Rightarrow$ no. of vertices in the second component.
⋮
$n_k \rightarrow$ no of vertices in the k^{th} component

Graph coloring

Coloring a graph constitutes coloring vertices / edges of the graph.

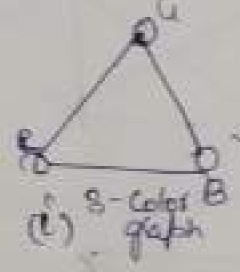
Coloring all the vertices of a graph is the property that no two adjacent vertices have same color.

Always try to color with minimum colors.

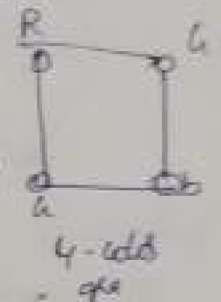
Chromatic numbers:

- It is defined as the least no. of colors needed for coloring the graph that denoted by $\chi(G)$.

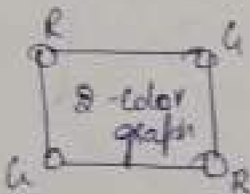
k-Chromatic graph



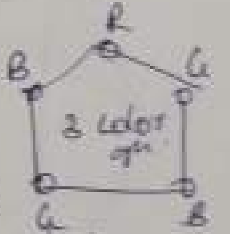
(i) 3-color graph



4-color graph



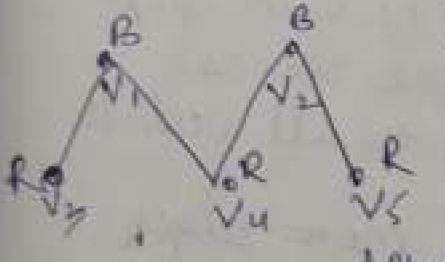
(iii)



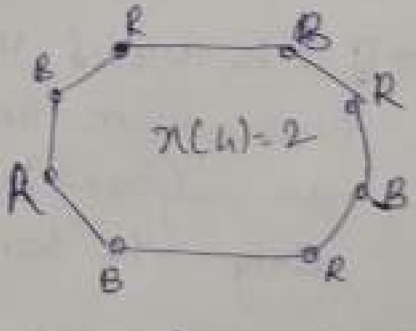
(iv)

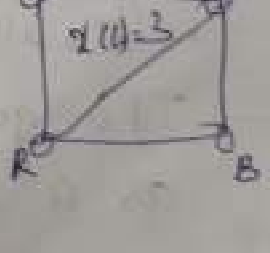
→ Every Bipartite graph is 2-colorable $\chi(G)=2$

$V = \{V_1, V_2, V_3, V_4, V_5\}$



= 2 coloring

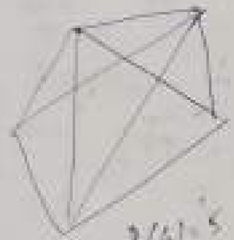




What is the chromatic no of following graph

$\chi(G) = 3$

$\chi(G) = 4$



χ

$\chi(G) = 5$

→ If it is isolated vertex then it is 1-chromatic because no two vertex of such graph are adjacent and therefore we can assign the same color to all vertices

→ A graph with one & more edges is atleast 2 chromatic

→ If a graph G contains a graph G_1 as a subgraph then $\chi(G) \geq \chi(G_1)$

→ If G is a graph G contains a graph G_1 as a subgraph

→ If G is a graph of n vertices then $\chi(G) \leq n$

→ $\chi(K_n) = n$ for all $n \geq 1$

→ If a graph G contains K_n as a subgraph then $\chi(G) \geq n$

Four colour problem

→ This has been proved every simple connected planar graph is properly colorable with 5 colors, here is a question that is it possible to adjust with 4 colors

→ This question was raised 1852 and was called the four colour problem

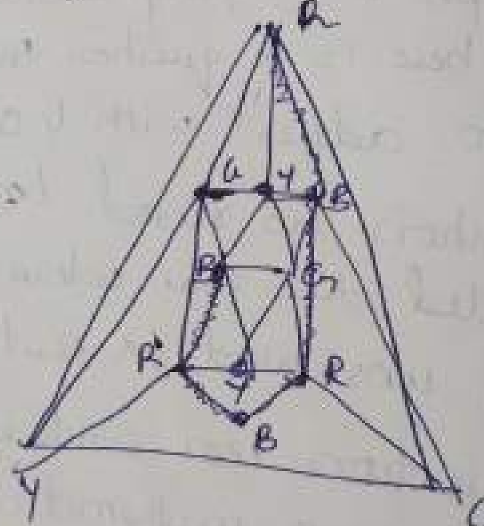
→ remains unsolved for centuries

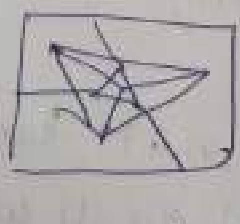
→ This question was eventually settled by two American mathematics, Kenneth Appel and Wolfgang Haken in 1976

→ After the several work on it they came out with the truth of the 4 color conjecture.

"Every simple, connected planar graph is 4 colorable"

- This 4-colour problem provided one of the original motivations for the development of algebraic graph theory.
- used in real world problems like minimising conflicts when scheduling sports events, planning examination timetables
- organising seating plans,
- CCTV camera placement in a building in order to not to overlap.
- Its an foundation for sudoku puzzle



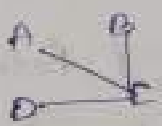


Spanning Tree and minimal spanning Tree

understand two graphs

1) undirected graph is a graph in which the edges do not point in any direction (bidirectional)

2) connected graph is a graph in which there is always a path from a vertex to any other vertex



spanning tree :

Let G be a connected graph. A spanning tree for subgraph T of G is called

(i) T is a tree

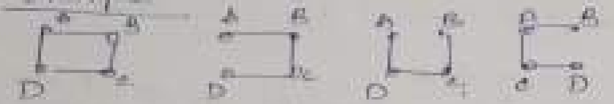
(ii) T include all vertices of G.

cyclic graph

subgraph / spanning tree

A spanning tree T of a graph is a subgraph of an undirected connected graph which includes all vertices of a graph with minimum possible number of edges. If a vertex is missed, then it is not spanning tree

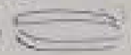
Example



minimal spanning tree :

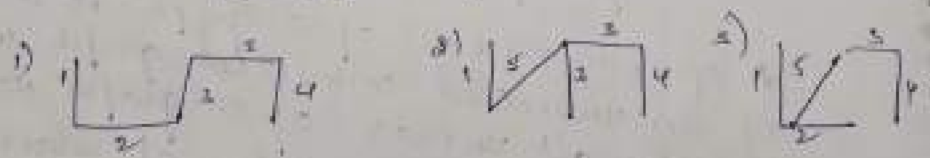
G be a graph and suppose there is positive real number associated with each edge of G. Then G is called a weighted graph and the positive real number associated with an edge E is called the weight of the edge e

Ex :- cities - road between two cities

city 1  city 2

A spanning tree whose weight is the least is called a minimal spanning tree of the graph.

· In all spanning tree they are
- weighted connected graph



① Kruskal's Algorithm

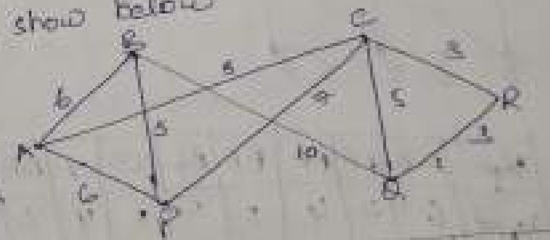
— working rule for the kruskal's method is called kruskal's Algorithm

Step 1 : Given a connected, weighted graph G with n vertices, list the edges of G in the order of non-decreasing weights

Step 2 : Starting from smallest weight edge, proceed sequentially by ~~not~~ selecting one edge at a time such that no cycle is formed

Step 3 : Stop the process of step 2 when n-1 edges are selected. These n-1 edges constitute a minimal spanning tree of G.

Example 1) using the kruskal's algorithm, find the a minimal spanning tree of the weighted graph show below



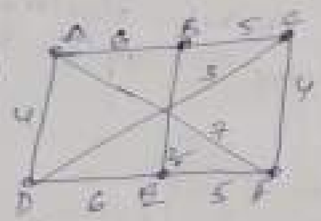
Edges	AB	AP	AC	BQ	BP	CP	CQ	CR	RQ
weights	6	6	8	10	5	7	5	3	3

Minimum cost ordered table

Edges	CE	BE	BF	EG	AD	FF	EF	DC	BG
weight	3	3	3	5	6	6	7	8	10
select	✓	✓	✓	>	✓	×	✓	×	×

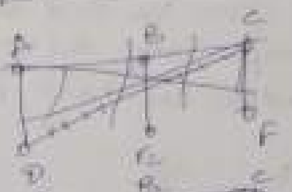
n = vertices = 6
n − 1 = edges = 5

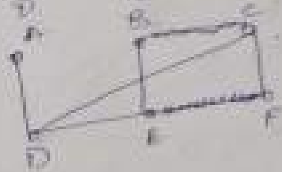
2)



Edges	AB	BF	AD	BE	BC	CD	CF	DE	EF
weight	6	7	4	3	5	3	4	6	5

Edges	BE	CD	CF	AD	BC	EF	AB	AF	DE
weight	2	3	4	4	5	5	6	7	6
selected	✓	✓	✓	✓	×	×	×	×	✓





n = 6
n − 1 = 5

Prim's Algorithm

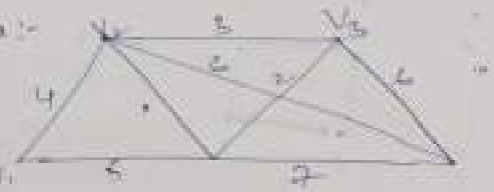
step 1: connected, weighted graph with n vertices, say A, B, C, … so on or $V_1, V_2, \ldots V_n$ vertices

- Preparing n×n table in which the weights of all edges
- no entries appear in diagonal
- Indicate the weights of the non-existing edges as ∞.

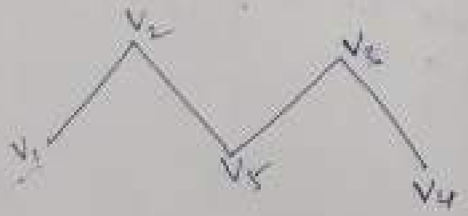
step 2 :- Connected to the nearest neighbour having smallest entry among all entries from v_1 and v_2 rows.

step 3 :- Start from v_m and repeat the process of step 2. Stop the process when all the n vertices can have n-1 edges.

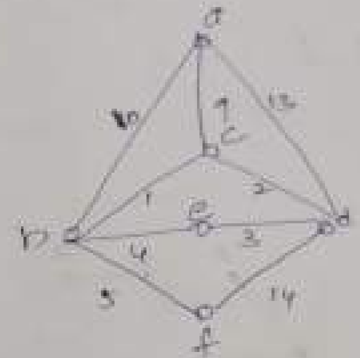
Ex:-



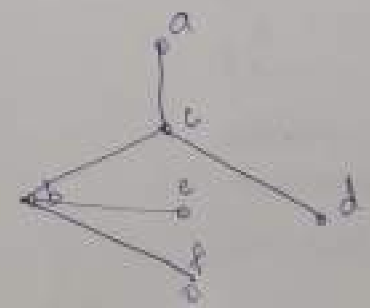
	v_1	v_2	v_3	v_4	v_5
v_1	–	4	∞	∞	5
v_2	4	–	3	6	1
v_3	∞	3	–	6	2
v_4	∞	6	6	–	7
v_5	5	1	2	7	–



2)



	a	b	c	d	e	f
a	–	10	9	13	∞	∞
b	10	–	1	∞	4	5
c	9	1	–	2	∞	∞
d	13	∞	2	–	3	14
e	∞	4	∞	3	–	∞
f	∞	5	∞	14	∞	–

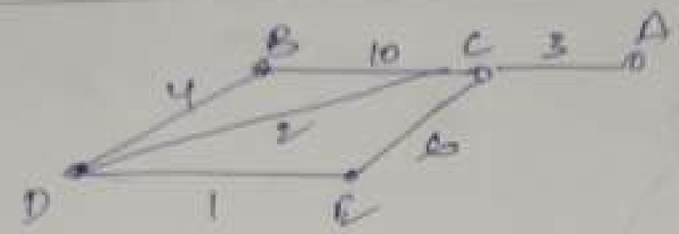


n vertices = 6

n-1 edges = 5

Hence it is proved as prim's spanning tree.

Scanned with ACE Scanner



	A	B	C	D	E
A	–	∞	3	∞	∞
B	∞	–	10	4	∞
C	3	10	–	2	6
D	∞	4	2	–	1
E	∞	∞	6	1	–

$n = 5$ vertices.

$n-1 = 4$ edges.

Hence it is spanning tree using Prim's algorithm.

Graph Traversal

Graph traversal is a technique used for searching a vertex in a graph. The graph traversal is also used to decide the order of vertices is visited in the search process. A graph traversal finds the edges to be used in the search process without creating loops. That means using graph traversal we visit all the vertices of the graph without getting into looping path.

There are two graph traversal techniques

1) BFS (Breadth first search)
2) DFS (Depth first search)

1) BFS

BFS traversal of a graph produces a spanning tree as final result. spanning Tree is a graph without loops. we use Queue data structure with maximum size of total number of vertices in the graph to implement BFS traversal.

— traversing or searching ... of graph data structures

— we visit all the nodes & vertices level by level and the travel completes when all the nodes are visited.

Steps to implement:

Step 1 : Define a Queue of size total no. of vertices in the graph.

Step 2 : select any vertex as starting point for traversal. Visit that vertex and insert it into the Queue.

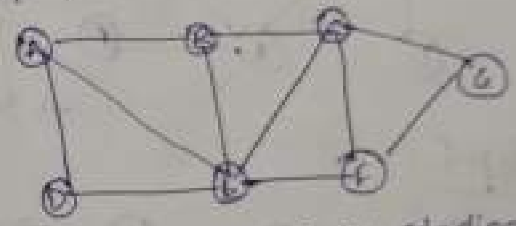
Step 3 : Visit all the non-visited adjacent vertices of the vertex which is at front of the Queue and insert them into the Queue

Step 4 — ~~Visit~~ when there is no new vertex to be visited from the vertex which is at front to the Queue then delete that vertex

Step 5 — Repeat steps 3 and 4 until queue becomes empty ~~step~~

Step 6 — when Queue become empty then produce final spanning tree by removing unused edges from the graph

Ex :- consider the following example graph to perform BFS traversal



Step 1 : Select the vertex A as starting point (visit A)

— insert A into the Queue

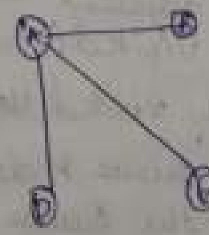
Queue

A					

Step 2 : visit all adjacent vertices of A i.e (D, E, B)

→ Insert visited vertices in Queue

D	E	B		



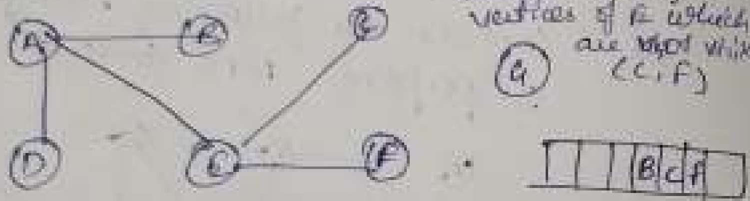
Step 3

← Visit all adjacent vertices of D (no vertex is present

→ Delete D from Queue

Step 4

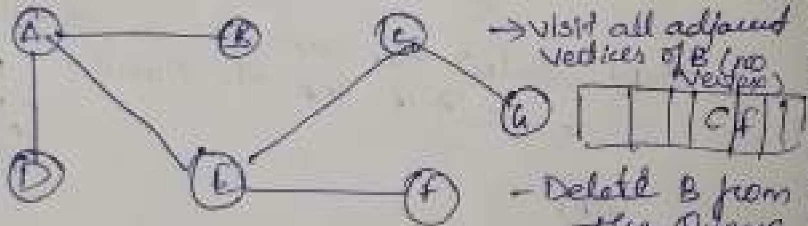
→ visit all adjacent vertices of E which are not visited (C, F)



→ Insert new vertex into the Queue and delete E

Step 5

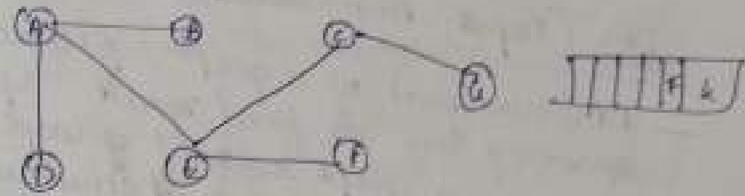
→ visit all adjacent vertices of B (no vertex)



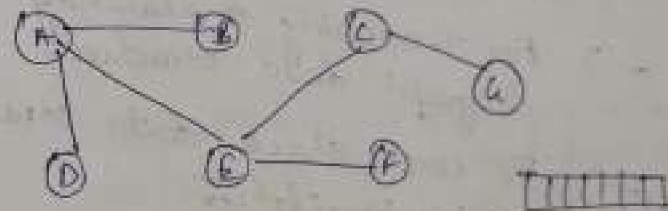
- Delete B from the Queue

Step 6

- Visit all adjacent vertices of C which are not visited (G)
- Insert newly visited vertex into the Queue & delete C from the Queue.



Step 7 and step 8



- Visit all adjacent vertices of F which are not visited (there is no vertex)
- Delete F from the Queue

- Visit all adjacent vertices of G which are not visited (there is no vertex)
- Delete G from the Queue.
- Queue become empty. So, stop the BFS Process

DFS (Depth First search)

— DFS traversal of a graph produces a spanning tree as final result. spanning tree is a graph without g loops. we use stack data a d structure with max. size of total no. of vertices in a graph to implement DFS traversal.

→ For traversing & searching a tree & graph data structure.

→ It uses stack data structure for implementation

→ In DFS one starts at the root and explores as far as possible along each branch before backtracking.

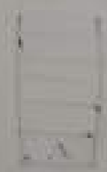
Step 1: Define a stack of size total no of vertices in the graph.

Step 2: Select any vertex as starting point for traversal. Visit that vertex and push it on to the stack.

Step 3: Visit any one of non-visited adjacent vertices of a vertex which is at the top of stack and push it on to the stack.

Step 4: Repeat 3 until there is no new vertex to be visited from the vertex which is at the top of the stack.

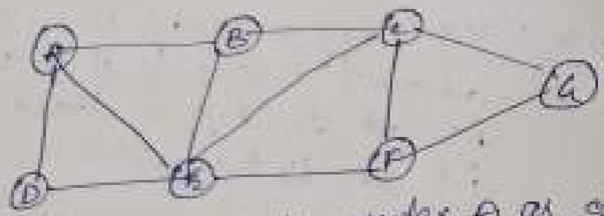
Step 5: When there is no new vertex to visit then we back tracking and pop one vertex from the stack.



Step 6: Repeat 3,4,5 until stack becomes empty.

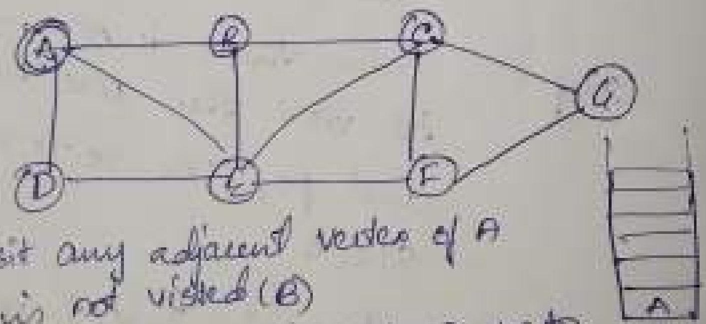
Step 7: when Stack becomes Empty, then produce final spanning tree by removing unused edges from the graph.

Sol:



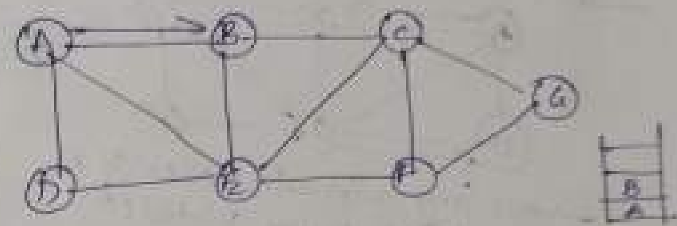
- Select the vertex A as starting Point (visit A)
- Push A on to the Stack

Step 1

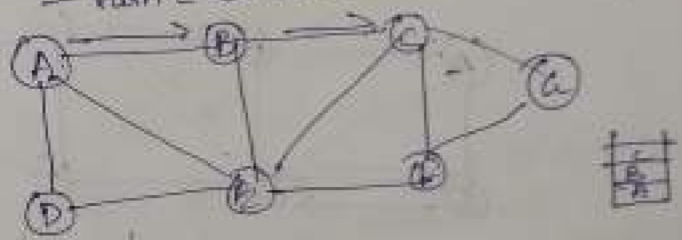


Step 2: - visit any adjacent vertex of A which is not visited (B)
- Push newly visited Vertex B onto to the stack

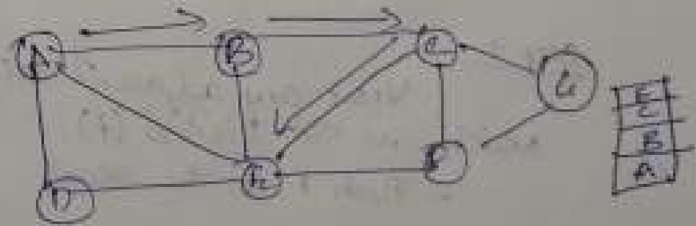
Step 2



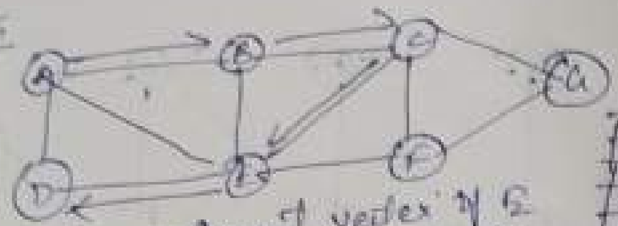
Step 3 → visit any adjacent vertex of B which is not visited.
- Push C on to the stack.

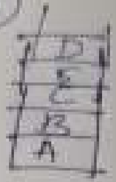


Step 4 Visit any adjacent vertex of C which is not visited E
- Push E onto the stack.



step 5

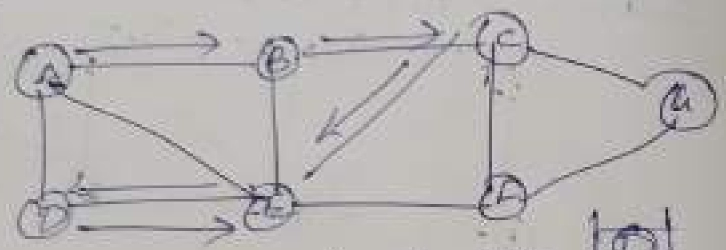


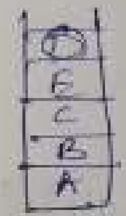


- visit any adjacent vertex of E which is not visited (D)

→ Push D onto stack.

Step 6



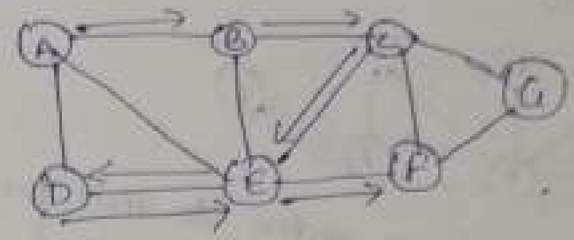


- There is no new vertex to visit from D. so we back track

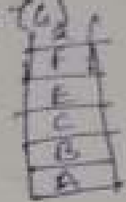
- Pop D from the stack

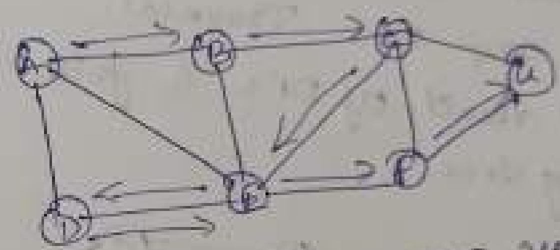
step 7 :- Visit any adjacent vertex of E which is not visited (F)

- Push F on the stack



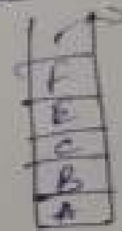
step 8: Visit any adjacent vertex of F (G)
Push G on to the stack

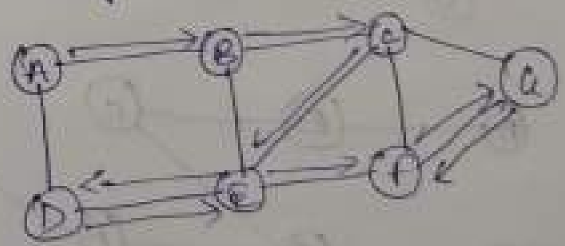




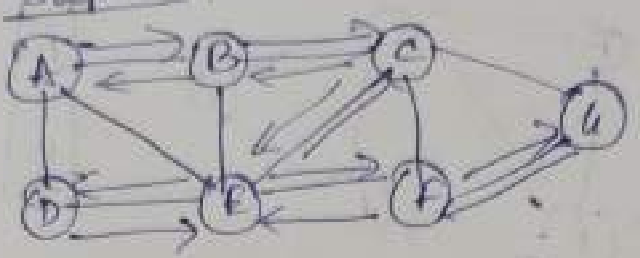
step 9: There is no new vertex to be visited from G, so we back track from the stack

— Pop G from the stack





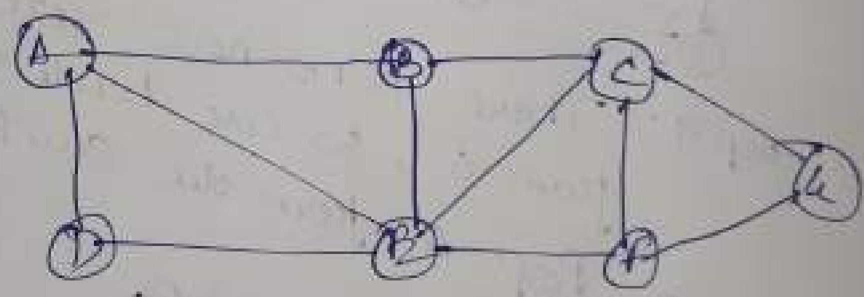
Step 10 :

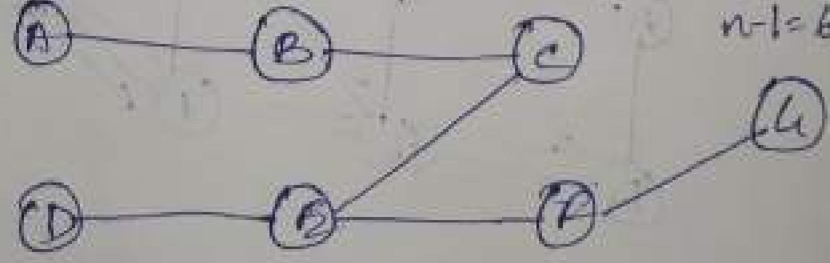


- There is no vertex to visit from E, C, B, A. So we ~~use~~ back track
- Pop E, C, B, A from the stack
- Stack become Empty.

So stop DFS Traversal.

Final result of DFS is following Spanning tree :





n = 7 Nodes
n-1 = 6 edges