

Unit - IV

Elementary Combinatorics

4.1: Basics of Counting.

4.2: Combinations and Permutations

4.3: Enumeration of Combinations and Permutations.

4.4: Enumerating Combinations and Permutations with Repetitions.

4.5: Enumerating Permutations with Constrained Repetitions.

4.6: Binomial Coefficient

4.7: The Binomial and Multinomial theorems.

4.8: The principle of Exclusion.

4.1! Basics of Counting:-

There are two basic rules of counting. They are.

① Sum Rule &

② Product Rule.

① SUM Rule:-

If a task can be done in ' n_1 ' ways and a second task in ' n_2 ' ways and if these two tasks cannot be done at the same time. then there are " $n_1 + n_2$ " ways to do either task.

Example problem:-

① The department will award a free computer to either a CS student or a CS professor. How many different choices are there if there are 530 students & 15 professors?

② There are $530 + 15 = 545$ choices.

② Suppose there are 16 boys and 18 girls in a class. How many ways to select one of these students either a boy or a girl as class representative

Sl: Given,

no. of boys = 16

no. of girls = 18

task-1: No. of ways for selecting a boy as a class representative
= n -ways $\Rightarrow$ 16 ways.

task-2: No. of ways for selecting a girl as a class representative
= n -ways $\Rightarrow$ 18 ways.

no. of ways of selecting a boy (or) a girl as a class representative = $16 + 18 = (n+m)$
 $= 34$ ways.

② PROBABILITY

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do 1st task and for each of these ways of doing the 1st task, there are n_2 ways to do the procedure.

Example:-

① How many different licence plates are there that containing exactly three english letters?

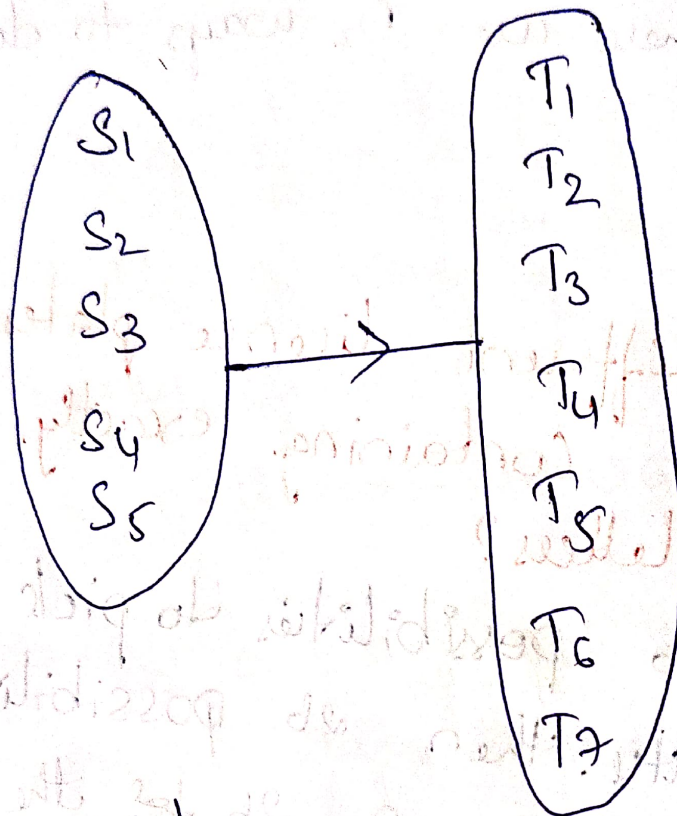
Sol: There are 26 possibilities to pick the first letter. then 26 possibilities for the second one & 26 for the last one.

∴ So, there are $26 \cdot 26 \cdot 26 = 17576$.

∴ 17576 different licence plate.

Q. Suppose a person has 5 shirts & 7 ties. How many ways a person can choose a shirt & Tie?

Given,
Shirts = 5
Ties = 7



n-Shirts.

m-Ties.

S₁ → Among 7 ties, he can choose one tie in 7 ways.

S₂ → 7 ways

S₃ → 7 ways

S₄ → 7 ways.

S₅ → 7 ways.

⇒ ~~7~~ m × n

⇒ 7 × 5

⇒ 35 ways.

Permutations :-

An arrangement in a sequence of elements of a set is called permutation of elements.

$${}^n P_r = \frac{n!}{(n-r)!}$$

→ It helps to determine how many different arrangements are possible.

Example problem

1) Suppose you have the letters A, B, C and you want to arrange them in a line of 2 positions.

Given,

Letters = {A, B, C}

the possible

arrangements would be.

- ① AB
- ② BA
- ③ AC
- ④ CA
- ⑤ BC
- ⑥ CB

So, there are 6 different permutations of 2 letters chosen from the set $\{A, B, C\}$

→ where ' n ' → is the total no. of objects.
' r ' → is the no. of positions to be filled.
' $n!$ ' → denotes the factorial of n ,

which is the product of all the integers from 1 to n .

In the example above,

$n=3$ (as there are 3 letters)

$r=2$ (as we are arranging them in 2 positions)

$${}_n P_r = \frac{n!}{(n-r)!}$$

$${}_3 P_2 = \frac{3!}{(3-2)!}$$

$$= \frac{3!}{1!}$$

$$= \frac{3 \times 2 \times 1}{1}$$

$${}_3 P_2 = 6$$

therefore, there are 6 permutations of 2 letters chosen from the set $\{A, B, C\}$.

2) 7 letters can be arranged in 4 spaces. $\{A, B, C, D, E, F, G\}$.

Q. Given,

$$7 \text{ letters} = \{A, B, C, D, E, F, G\}$$

$$n = 7,$$

$$r = 4,$$

$${}_n P_r = \frac{n!}{(n-r)!}$$

$$= \frac{7!}{(7-4)!}$$

$$= \frac{7!}{3!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= 840 //$$

Combinations:

An ordered Selection of r elements of set containing n distinct element is called an n -combinations of n elements and is denoted by $C(n, r)$ (or) nC_r (or) nC_r .

$${}^nC_r = \frac{n!}{(n-r)! \cdot r!}$$

$${}^nC_n = 1$$
$${}^nC_0 = 1$$

Ex ${}^{10}C_2 = \frac{10!}{(10-2)! \cdot 2!}$

$$= \frac{10!}{8! \cdot 2!}$$

$$= \frac{10 \times 9 \times 8!}{8! \cdot 2!}$$

$$= \frac{5 \times 10 \times 9}{2 \times 1}$$

$${}^{10}C_2 = 45 //$$

Example 1
1) Find the value of 'n' if ${}^nC_{n-2} = 10$.

Given,

$${}^nC_{n-2} = 10.$$

where $r = n-2$.

$${}^nC_r = \frac{n!}{(n-r)! r!}$$

$$\Rightarrow \frac{n!}{(n-(n-2))! (n-2)!} = 10.$$

$$\Rightarrow \frac{n!}{(n-n+2)! (n-2)!} = 10.$$

$$\Rightarrow \frac{n!}{2! (n-2)!} = 10.$$

$$\Rightarrow \frac{n(n-1)(n-2)!}{2! (n-2)!} = 10.$$

$$\Rightarrow \frac{n^2 - n}{2} = 10$$

$$\Rightarrow n^2 - n = 20$$

$$\Rightarrow n^2 - n - 20 = 0.$$

$$\Rightarrow n^2 - 5n + 4n - 20 = 0.$$

$$\Rightarrow n(n-5) + 4(n-5) = 0.$$

$$\Rightarrow (n-5)(n+4) = 0.$$

$$\Rightarrow n-5=0$$

$$n=5$$

$$n+4=0$$

$$n=-4.$$

$\therefore$ In the combination, no need to take 've' value.

$\therefore n$ value is 5.

② A committee of 5 people is to be formed a group of 4 men & 7 women. How many possible committees can be formed if at least 3 women are in the committee.

Given,

total no. of committee = 5

Men = 4

Women = 7.

$$\text{No. of committees} = {}^7C_3 {}^4C_2 + {}^7C_4 {}^4C_1 + {}^7C_5 {}^4C_0$$

$$\frac{4!}{(4-3)!3!} \cdot \frac{4!}{(4-2)!2!} + \frac{4!}{(4-1)!1!} \cdot \frac{4!}{(4-0)!0!} + \frac{4!}{(4-0)!0!}$$

$$\Rightarrow \frac{4!}{4!3!} \cdot \frac{4!}{2!2!} + \frac{4!}{3!4!} \cdot \frac{4!}{3!1!} + \frac{4!}{2!5!} \cdot \frac{4!}{4!}$$

$$\Rightarrow \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{\cancel{3 \times 2 \times 1}} \cdot \frac{1}{4!} + \frac{\cancel{7 \times 6} \times 5 \times 4 \times 3 \times 2!}{\cancel{2 \times 1}} \cdot \frac{1}{\cancel{3 \times 2 \times 1}} + \frac{7 \times 6 \times 5!}{2! \cancel{4!}}$$

$$\Rightarrow 210 + 160 + 210$$

$$\Rightarrow 371$$

Problems for Permutations & Combinations

1) Find the number of distinguishable permutations of the letters in the following words:

1) DISCRETE MATHEMATICS.

2) ENGINEERING.

3) CALCULUS.

4) STRUCTURES.

5) PASCAL.

Sol (1) Given:

DISCRETE MATHEMATICS.

→ It has 19 letters, out of this.

we have,

D-1

I-2

S-2

E-2

R-1

T-3

M-2

A-2

H-1

H-1

∴ the total no. of permutations of the letters of word DISCRETE MATHEMATICS

$$n = 19$$

is

$$19!$$

$$\frac{3! \cdot 3! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!}{9 \cdot 4 \cdot 4 \cdot 2}$$

$$9, 16, 2$$

⇒

$$3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 4 \times 2 \times 1$$

$$19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11$$

⇒

$$2 \times 6 \times 6 \times 2 \times 1 \times 1$$

$$\Rightarrow 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 10 \times 9 \times 7 \times 5 \times 2 \times 1$$

$$19!$$

$$\frac{19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{9 \times 16 \times 2}$$

$$11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11$$

$$11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11 \times 11$$

DISCRETE = 8

D-1

I-1

S-1

C-1

R-1

E-2

T-1

8!

1! 1! 1! 1! 1! 1! 1! 2! 1!

$\Rightarrow 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

~~2 \times 1~~

$\Rightarrow 20,160$

MATHEMATICS = 11

M-2

A-2

T-2

H-1

E-1

I-1

C-1

S-1

11!

2! 2! 2! 1! 1! 1! 1! 1! 1!

$\Rightarrow 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

~~2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 1 \times 1 \times 1 \times 1~~

$\Rightarrow 4,989,600$

Q ENGINEERING.

It has 11 letters. out of this,
we have,

E - 3

N - 3

G - 2

I - 2

R - 1

the total no. of permutations of the
letters of word ENGINEERING.

$$\Rightarrow \boxed{n = 11.}$$

11!

$\Rightarrow$

$$\frac{11!}{3! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!}$$

$$\Rightarrow \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1 \times 1 \times 1 \times 1}$$

$$\Rightarrow 11 \times 10 \times 9 \times 8 \times 7 \times 5.$$

$$\Rightarrow 2,77,200 \text{ ways.} //$$

③ CALCULUS:-

It has 8 letters out of this
we have,

C-2

A-1

L-2

U-2

S-1

∴ the total no. of permutations of the
letter of word CALCULUS

$$n = 8$$

$$\Rightarrow \frac{8!}{2! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 1!}$$

$$\Rightarrow \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 1 \times 1}$$

$$\Rightarrow 5,040 \text{ ways.}$$

⑥ STRUCTURES:-

It has 10 letters out of this
we have,

S - 2

T - 2

R - 2

U - 2

C - 1

∴ the total no. of permutations of
the letters of word STRUCTURES.
 $n = 10.$

$10!$

$$\Rightarrow \frac{10!}{2! 2! 2! 2! 1!}$$

$$\Rightarrow \frac{10 \times 9 \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{2} \times 1 \times \cancel{2} \times 1 \times \cancel{2} \times 1 \times 1 \times 1}$$

$$\Rightarrow 226,800 \text{ ways.}$$

5) PASCAL :-

It has 6 letters out of this
we have.

P-1

A-2

S-1

C-1

L-1

∴ the total no. of permutations of the
letter of word PASCAL.

$$n = 6$$

$$\Rightarrow \frac{6!}{2! 1! 1! 1! 1! 1!}$$

$$\Rightarrow \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 1 \times 1 \times 1 \times 1}$$

⇒ 360 ways //

Problem ② - In how many ways can 6 men & 6 women be seated in a row:

i) if any person may sit next to any other?

ii) if men and women must occupy alternate seats?

Given,

$$\text{Men} = 6$$

$$\text{Women} = 6$$

i) If any person may sit next to any other, no distinction need to be made between men and women in their seating.

Accordingly, since there are 12 persons in all, the number of ways they can be seated is.

$$\underline{M} \underline{M} \underline{M} \underline{M} \underline{M} \underline{M} = 6$$

$$\underline{W} \underline{W} \underline{W} \underline{W} \underline{W} \underline{W} = 6$$

$$\underline{12}$$

$$n = 12! \Rightarrow 4,79,001,600$$

ii) When men & women are to occupy alternate seats, the six men can be seated in $6!$ ways in odd places and the six women can be seated in $6!$ ways in even places, & corresponding to each arrangement of the men there is an arrangement of the women.

→ therefore, the number of ways in which the men occupy the odd places & the women the even places is

M W M W M W M W M W M W

$$\Rightarrow m = 6!$$

$$w = 6!$$

$$\Rightarrow m \times w$$

$$\Rightarrow 6! \times 6! = 720 \times 720$$

$$\Rightarrow 518400$$

$$\Rightarrow 518400 //$$

∴ Similarly, the no. of ways in which the women occupy the odd places & the men occupy the even places is

W M W M W M W M W M W M

$$\Rightarrow m=6!$$

$$\Rightarrow n=6!$$

$$\Rightarrow m \times n$$

$$\Rightarrow 6! \times 6!$$

$$\Rightarrow 720 \times 720$$

$$\Rightarrow 518400.$$

$\therefore$ Accordingly, the total no. of ways is

$$518400 + 518400 = 1,036,800.$$

$\therefore 1,036,800$ ways.

③ A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations.

- i) there is no restriction on the choice.
- ii) two particular persons will not attend separately.
- iii) Two particular persons will not attend together.

Sol/ Given,

relatives = 11.

Invites = 5.

i) Since, there is no restriction on the choice of invitees, five out of 11 can be invited in

$$C(11, 5) = \frac{11!}{(11-5)!5!} = 462 \text{ ways.}$$

ii) Since two particular persons will not attend separately,

→ they should both be invited & not invited.

→ If both of them are invited, then three more invitees are to be selected from the remaining 9 relatives.

$$\begin{array}{l} \text{Total} = 11 \\ \text{Invited} = 2 \end{array} \} \Rightarrow 9$$

↓

$$\text{more to be selected} = 3$$

→ This can be done in

$$\begin{aligned} C(9, 3) &= \frac{9!}{(9-3)!3!} \\ &= \frac{9!}{6!3!} \Rightarrow 84 \text{ ways} \end{aligned}$$

$$\boxed{n_1 = 84}$$

2) If both of them are not invited, then five invitees are to be selected from 9 relatives, this can be done in

$${}^9C_5 = \frac{9!}{(9-5)! \cdot 5!}$$

$$\Rightarrow \frac{9!}{4! \cdot 5!} \Rightarrow \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = 126 \text{ ways}$$

$$n_2 = 126$$

therefore, the total no. of ways in which the invitees can be selected in this case is

$$n_1 + n_2 \Rightarrow 84 + 126 = 210$$

ii) Since two particular persons (say A & B) will not attend together, only one of them can be invited or none of them can be invited, the number of ways of choosing the invitees with A invited is

$${}^9C_4 = \frac{9!}{(9-4)! \cdot 4!} \Rightarrow \frac{9!}{5! \cdot 4!} = 126 \Rightarrow n_1$$

Similarly, the no. of ways of choosing the invitees with B invited is 126. $\rightarrow n_2$

$\Rightarrow$ If both A & B are not invited, the no. of ways of choosing the invitees is

$$C(9, 5) = \underline{126} \rightarrow n_3$$

$\rightarrow$ Thus, the total no. of ways in which the invitees can be selected in this

case is $\rightarrow \boxed{n_1 + n_2 + n_3}$

$$126 + 126 + 126 = 378 //$$

Permutation with Repetition

① If 'p' represents no. of different permutations of n things taken all at a time, when p of them are of one kind, q of them another kind, & r of third kind & so on. then.

$$\boxed{P = \frac{n!}{p! \cdot q! \cdot r!}}$$

Permutation with Repetition

Permutation with Repetition

involves arranging elements where duplicates are allowed.

In simpler terms, it's the arrangement of items where the same item can appear multiple times in a sequence.

For instance, in the set $\{A, B, C\}$, a permutation with repetition might include sequences like AA, AB, AC, BB etc.

The total no. of permutations with repetitions can be calculated using

the formula n^r , where n is the

no. of options for each position, &

r is the length of the permutation.

Ex: How many 4 digit no. can be formed from six digits repetition is not allowed. $(1, 2, 3, 5, 7, 8)$.

Given digits $\Rightarrow 1, 2, 3, 5, 7, 8$.

$$\Rightarrow n = 6$$

$$r = 4$$

Repetition is not allowed.

for digit number $\Rightarrow nP_r = {}^6P_4$

$$\Rightarrow \frac{6!}{(6-4)!}$$

$$= \frac{6!}{2!} \Rightarrow \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

= 360 ways

2) How many arrangements can be made with letters of the word

EXAMINATION,

Sol: Given word is EXAMINATION,

total letters in the given word is

= 11.

$$P = \frac{n!}{p! q! r!}$$

repeated letters are

A = 2

I = 2

N = 2

$$P = \frac{11!}{2! 2! 2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1}$$

$$= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

$$= 49,89,600 //$$

Conditions for Permutation with Repetition

Case-1: When there is no restrictions

Case-2: When all the words begin & ends with particular letters.

Case-3: When few letters [vowels / consonants] occur together.

Case-4: When vowels consonants occupy odd / even place.

$$\text{total no. of ways} = \frac{n!}{x! y! z!}$$

Find the no. of permutations that can be made out of the letters.

MISSISSIPPI

Case-1: MISSISSIPPI

total letter - 11

M - 1

I repeated in 4 time

S $\rightarrow$ 4

P - 2

total no. of permutations

$$= \frac{11!}{2!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2}$$

$$= \frac{4! \times 4! \times 2! \times 1!}{2}$$

$$= 34650$$

Case-iii:- All 'S' together.

SSSS $\rightarrow$ 1 digit M I P P P $\rightarrow$ 4 digits

$4 + 1 = 5$ digits.

$Q = 4$

$P = 2$

total no. of permutations = $\frac{8!}{2!}$

$$= \frac{8!}{2!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2}$$

$$= 4 \times 7 \times 6 \times 5$$

≥ 840 ways

Q) In how many ways can the letters of the word HEXAGON be arranged so that the vowels are always in even place.

Sol: The word HEXAGON has 7 letters of which 3 are vowels.

Since total letters is 7, there are

4 odd &

3 even places -

the 3 vowels can be arranged in the 3 odd places in $3!$ ways.

for each of this 4 consonants can be arranged in 4 odd places in $4!$ ways.

hence, total ways = $3! \times 4!$

$$= 6 \times 24$$

$$= 144 \text{ ways.}$$

Q. Prove that, for all integers $n, r \geq 0$,
if $n+1 > r$, then $P(n+1, r) = \left(\frac{n+1}{n+1-r} \right) P(n, r)$.

Sol: Given,

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(n+1, r) = \frac{(n+1)!}{(n+1-r)!}$$

therefore,

$$\frac{P(n+1, r)}{P(n, r)} = \frac{(n+1)!}{(n+1-r)!} \times \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{(n-r)!}{(n+1-r)!}$$

$$= \frac{n+1}{n+1-r}$$

$$\text{So that, } P(n+1, r) = \left(\frac{n+1}{n+1-r} \right) P(n, r).$$

→ Combinations with Repetitions

Suppose we wish to select, with repetition, a combination of r objects from a set of n distinct objects. The no. of such selections is given by.

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

In other words,

$$C(n+r-1, r) = C(n+r-1, n-1)$$

n - distinct objects.

r - Identical objects with repetitions.

∴ No. of such selections are

$$C(n+r-1, r) = C(n+r-1, n-1)$$

The following are the other interpretations of this number.

① $C(n+r-1, r) = C(n+r-1, n-1)$ represents the no. of ways in which r identical objects can be distributed among n

distinct containers.

② $C(n+r-1, r) = C(r+n-1, n-1)$ represents the no. of nonnegative integer solutions of the eqⁿ.

$$x_1 + x_2 + \dots + x_n = r.$$

Problem:

i) In how many ways can we distribute 10 identical marbles among 6 distinct containers?

Solⁿ Given,

10 identical marbles is

$$r = 10$$

6 distinct ~~marble~~ containers

$$\text{all } n \geq 6 \Rightarrow n + r = (r, 1 - r + n)$$

no. of such selections = $C(r+n-1, r)$

$$= C(10+6-1, 10)$$

$$\text{therefore } (1 - n, 1 - n + r) \geq C(15, 10)$$

$$\frac{(15 - 10)!}{(10 - 10)!} \Rightarrow \frac{15!}{5!}$$

$$= \frac{15!}{10! 5!} =$$

$$\Rightarrow \frac{3 \times \cancel{15} \times \cancel{14} \times 13 \times \cancel{12} \times \cancel{11} \times \cancel{10}}{10! \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$\Rightarrow 3 \times 7 \times 13 \times 11.$$

$$\Rightarrow 3003_{11}$$

Find the no. of non-negative integers solutions of the eqⁿ $x_1 + x_2 + x_3 + x_4 + x_5$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

Solⁿ Given,

x_1, x_2, x_3, x_4, x_5 are the

non-negative integers.

Sum of this integers are $= 8 = r$.

$$n = x_1, x_2, x_3, x_4, x_5 = 5$$

$n = 5$ (no. of non-negative integers) = 5

$$r = 8$$

$$C(n+r-1, r)$$

$$= C(5+8-1, 8)$$

$$= C(12, 8)$$

$$.01$$

$$\rightarrow \frac{(10-2)1.21}{121}$$

$$\rightarrow \frac{121}{(12-8)181}$$

$$= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

$$= 81 \times 10 \times 11 \times 12$$

$$= 12 \times 11 \times 10 \times 9$$

$$12 \times 11 \times 10 \times 9$$

$$=$$

$$= 495$$

$$=$$

$$= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

Binomial & Multinomial Theorem

Binomial Theorem:

A binomial theorem describes the algebraic expansion of powers of a binomial with 2 variables.

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

it can be written as:

$$(x+y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$$

It can also be written as

$$C(n, x) \equiv \binom{n}{x}$$

$$\Rightarrow x^r y^{n-r}$$

$$(x+y)^n = \sum_{x=0}^n \binom{n}{x} x^r y^{n-r}$$

$$\Rightarrow (1+x)^n = \binom{n}{0} 1 \cdot x^0 + \binom{n}{1} 1^{n-1} \cdot x^1 + \binom{n}{2} 1^{n-2} \cdot x^2 + \dots$$

$$\binom{n}{3} 1^{n-3} \cdot x^3 + \dots$$

$$= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots$$

$$\boxed{\text{b=1}}$$

$$= 1x^0 + \frac{n!}{(n-1)!1!} x^1 + \frac{n!}{(n-2)!2!} x^2 + \frac{n!}{(n-3)!3!} x^3 + \dots$$

$$= 1 + \frac{n(n-1)!}{(n-1)!1!} x + \frac{n(n-1)(n-2)!}{(n-2)!2!} x^2 + \dots$$

$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!3!} x^3 + \dots$$

$$(1+x)^n \Rightarrow 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Problem:

1) Find the coefficient of

i) $x^9 y^3$ in the expansion of $(x+2y)^{12}$.

ii) $x^5 y^2$ in the expansion of $(2x-3y)^7$.

Sol 1) W.K.T Binomial theorem.

$$(x+y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$$

So the binomial theorem.

Adding

$$x = x, \quad y = 2y, \quad n = 12.$$

$$(x+2y)^{12} = \sum_{r=0}^{12} {}^{12}C_r x^{12-r} (2y)^r$$

$$= \sum_{r=0}^{12} {}^{12}C_r x^{12-r} 2^r y^r$$

$$= \sum_{r=0}^{12} {}^{12}C_r 2^r x^{12-r} y^r \quad \text{--- (1)}$$

We have to find out $x^9 y^3$.
Term can be compared with

$$x^9 y^3$$

eqn (1) is

$$x^9 y^3 = x^{12-r} y^r$$

$$x^9 = x^{12-r}$$

$$y^3 = y^r$$

Since the eqn are equal = Powers should be equal

$$9 = 12 - r$$

$$3 = r$$

$$r = 12 - 9$$

$$r = 3$$

∴ Substitute $r = 3$ in eq ① is

$$\Rightarrow \sum_{r=0}^{\infty} {}^{12}C_r \cdot 2^3 \cdot x^{12-r} \cdot y^3$$

$$\Rightarrow {}^{12}C_3 \cdot 2^3 \cdot x^9 \cdot y^3$$

$$\Rightarrow \text{Reqd } 2$$

As we have to find coefficient of $x^9 y^3$.

we consider.

$$\Rightarrow {}^{12}C_3 \cdot 2^3 \cdot x^9 \cdot y^3$$

$$\Rightarrow \cancel{12 \times 11 \times 10}$$

$$\Rightarrow \frac{12!}{(12-3)! \cdot 3!} \cdot 2^3$$

$$\Rightarrow \frac{2 \times 12 \times 11 \times 10 \times 9!}{9! \cdot 3 \times 2 \times 1} \cdot 2^3$$

$$\Rightarrow 220 \times 2$$

$$\Rightarrow 1760$$

∴ the coefficient of $x^9 y^3$ in $(x+2y)^{12}$ is 1760.

ii) $x^5 y^2$ is the expansion of $(2x-3y)^7$.

Ans - 6048,

* Multinomial theorem:-

Multinomial theorem is a generalization of the binomial with more than 2 variables.

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{n_1, n_2, \dots, n_k} \frac{n!}{n_1! \times n_2! \times \dots \times n_k!}$$

where $n_1 + n_2 + \dots + n_k = n$.

the expression $\frac{n!}{n_1! n_2! \dots n_k!}$ is also written as.

$\binom{n}{n_1, n_2, n_3, \dots, n_k}$ is also called as

Multinomial coefficient.

Ex - Compute the following by using Multinomial theorem.

(a) $\binom{7}{2, 3, 2}$

(b) $\binom{4}{1, 1, 2}$

Ex (a) It is in the form of $\binom{n}{n_1, n_2, n_3}$

where $n_1 + n_2 + n_3 = n$

by applying Multinomial theorem.

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{7!}{2! \cdot 3! \cdot 2!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1}$$

$$= 210 //$$

⑥ According to Multinomial theorem

$$\frac{4!}{1! \cdot 1! \cdot 2!} = \frac{4 \times 3 \times 2 \times 1}{1 \times 1 \times 1 \times 2 \times 1}$$

$$= 12 //$$

3) Determine the coefficient of

① xyz^2 in the expansion of $(2x-y-z)^4$

Sol: By applying multinomial theorem.

$$(x_1 + x_2 + \dots + x_k)^n = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!} \cdot (x_1)^{n_1} \cdot (x_2)^{n_2} \cdot \dots \cdot (x_k)^{n_k}$$

where $n_1 + n_2 + \dots + n_k = n$.

the given expression is $(2x-y-z)^4$.

where
Apply

$n=4, x_1=2x, y, x_2=-y, x_3=-z$
multinomial theorem.

$$\frac{n!}{n_1! n_2! n_3!} \cdot (x_1)^{n_1} \cdot (x_2)^{n_2} \cdot (x_3)^{n_3}$$

$$= \frac{4!}{n_1! n_2! n_3!} \cdot (2x)^{n_1} \cdot (-y)^{n_2} \cdot (-z)^{n_3} \quad \text{--- (1)}$$

we have to find out the coefficient of xyz^2 is .

$$xyz^2 = x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$$

$$\boxed{x_1^{n_1} x_2^{n_2} x_3^{n_3} = xyz^2}$$

where $n_1=1, n_2=1, n_3=2$.

substitution of n_1, n_2, n_3 values in eq (1).

$$= \frac{4!}{1! \cdot 1! \cdot 2!} (2x)^1 \cdot (-y)^1 \cdot (-z)^2$$

$$= \frac{4 \times 3 \times 2 \times 1}{1! \times 1! \times 2 \times 1} \cdot 2x^1 \cdot (-y)^1 \cdot (z)^2$$

$$= 12 \cdot 2x(-y)z^2$$

$$= -24xyz^2$$

$$\boxed{xyz^2 = -24}$$

Combination with repetition.

1) In how many ways can we distribute 12 identical pencils to 5 children so that every child gets at least one pencil?

Self Given,

no. of pencils = 12

no. of children = 5

Every child gets at least one pencil.

→ each child may get ≥ 1 pencil.

→ So, if we distribute one pencil to each child then the remaining identical pencils $\Rightarrow (12 - 5 = 7)$.

can be distributed to 5 children.

→ no. of ways to distribute 7 identical pencils to 5 children is

$C(r+n-1, r)$; where $r=7$
 $n=5$

$$\Rightarrow C(7+5-1, 7)$$

$$\Rightarrow C(11, 7)$$

$$\Rightarrow \frac{11!}{7!(11-7)!}$$

$$\Rightarrow \frac{11!}{7! \cdot 4!}$$

$$\Rightarrow \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7! \cdot 4 \times 3 \times 2 \times 1}$$

$\Rightarrow 330$ ways.

the principle of Inclusion - Exclusion (27) :

→ the principle of Inclusion - Exclusion can be stated in terms sets as follows

→ let A_1 and A_2 be two sets, let T_1 be the tasks of choosing an element from A_1 , and T_2 be the tasks of choosing another element from A_2 .

→ there are no. of ways to do task T_1 or task T_2 is

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$

where $n(A_1 \cup A_2)$ is the no. of ways to do either tasks T_1 or T_2

$n(A_1 \cap A_2)$ is the no. of ways to do both tasks T_1 and T_2

→ let us take three tasks T_1, T_2 and T_3 let A_1, A_2 and A_3 are three sets. then no. of ways to do this tasks i.e. T_1 or T_2 or T_3 is

$$n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$$

Ex: Consider a set of Integers from 1 to 250 ^{find} ⁽²⁰⁾
 out

- ① How many of these numbers are divisible by 3 or 5 or 7
- ② " " " " " 5 or 7 1 to 250
- ③ " " " " " 3 or 7 1 to 250
- ④ " " " " " 3 or 5 or 7
- ⑤ " " " " " 3 or 5 but not by 7
- ⑥ " " " " " 5 or 7 but not by 3
- ⑦ " " " " " 3 or 7 but not by 5
- ⑧ How many of these Integers between 1 and 250 not divisible by 3, 5 and 7.

Sol: let A_1, A_2, A_3 are three sets consisting of Integers that are divisible by 3, 5 and 7 respectively from 1 to 250.

$$n(A_1) = \frac{250}{3} = 83 \rightarrow (\text{taking only Integer part})$$

$$n(A_2) = \frac{250}{5} = 50$$

$$n(A_3) = \frac{250}{7} = 35$$

~~$$n(A_1 \cap A_2) = \frac{250}{3 \times 5} = 16$$~~

~~$$n(A_2 \cap A_3) = \frac{250}{5 \times 7} = 7$$~~

$$n(A_1 \cap A_3) = \frac{250}{3 \times 7} = 11$$

$$n(A_1 \cap A_2 \cap A_3) = \frac{250}{3 \times 5 \times 7} = 2$$

(29)

$$n(A_1) = 83, n(A_2) = 50, n(A_3) = 35, n(A_1 \cap A_2) = 16, \\ n(A_2 \cap A_3) = 7, n(A_1 \cap A_3) = 11, n(A_1 \cap A_2 \cap A_3) = 2$$

① Ans: $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$
 $= 83 + 50 - 16$
 $= 117$

② Ans: $n(A_2 \cup A_3) = n(A_2) + n(A_3) - n(A_2 \cap A_3)$
 $= 50 + 35 - 7$
 $= 78$

③ Ans: $n(A_1 \cup A_3) = n(A_1) + n(A_3) - n(A_1 \cap A_3)$
 $= 83 + 35 - 11$
 $= 107$

④ Ans: $n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$
 $= 83 + 50 + 35 - 16 - 7 - 11 + 2$
 $= 136$

$$\begin{aligned} \textcircled{5} \text{ Ans: } &= n(A_1 \cup A_2 \cup A_3) - n(A_3) \\ &= 136 - 35 \\ &= 101 \end{aligned}$$

no. of Integers that are divisible by 3 or 5 but not by 7

$$\begin{aligned} \textcircled{6} &= n(A_1 \cup A_2 \cup A_3) - n(A_1) \\ &= 136 - 82 \\ &= 53 \end{aligned}$$

$$\begin{aligned} \textcircled{7} &= n(A_1 \cup A_2 \cup A_3) - n(A_2) \\ &= 136 - 50 \\ &= 86 \end{aligned}$$

$$\begin{aligned} \textcircled{8} \text{ no. of Integers that are not divisible by 3 or 5 or 7} \\ n(\overline{A_1 \cup A_2 \cup A_3}) &= \text{Total no. of Integers} - n(A_1 \cup A_2 \cup A_3) \\ &= 250 - 136 \\ &= 114 \end{aligned}$$

Ex: Consider a set of Integers from 0 to 499

- ① no. of Integers that are divisible by 3 or 5
- ② " " " " " 5 or 11
- ③ " " " " " 3 or 11
- ④ " " " " " 3 or 5 or 11
- ⑤ " " " " " 3 or 5 but not by 11

No. of Integers are divisible by 5 or 11 but not by 3
 " " " " " 3 or 11 but not by 5
 " " " " " not divisible by 3 or 5 or 11

Sol: Let A_1, A_2, A_3 are three sets containing the set of Integers that are not divisible by 3 or 5 and 11 respectively, from 0 to 499

$$n(A_1) = \frac{499}{3} = \frac{500}{3} = 166$$

$$n(A_2) = \frac{499}{5} = \frac{500}{5} = 100$$

$$n(A_3) = \frac{499}{11} = \frac{500}{11} = 45$$

$$n(A_1 \cap A_2) = \frac{500}{3 \times 5} = 33$$

$$n(A_2 \cap A_3) = \frac{500}{5 \times 11} = 9$$

$$n(A_1 \cap A_3) = \frac{500}{3 \times 11} = 15$$

$$n(A_1 \cap A_2 \cap A_3) = \frac{500}{3 \times 5 \times 11} = 3$$

$$n(A_1) = 166, n(A_2) = 100, n(A_3) = 45, n(A_1 \cap A_2) = 33$$

$$n(A_2 \cap A_3) = 9, n(A_1 \cap A_3) = 15, n(A_1 \cap A_2 \cap A_3) = 3$$

$$\begin{aligned}
 \text{Ans: } n(A_1 \cup A_2) &= n(A_1) + n(A_2) - n(A_1 \cap A_2) \\
 &= 166 + 100 - 33 \\
 &= 233
 \end{aligned}$$

2 Ans: $n(A_2 \cap A_3) = n(A_2) + n(A_3) - n(A_2 \cap A_3)$
 $= 100 + 45 - 9$
 $= 145 - 9$
 $= 136$

(32)

3 Ans: $n(A_1 \cap A_3) = n(A_1) + n(A_3) - n(A_1 \cap A_3)$
 $= 166 + 45 - 15$
 $= 211 - 15$
 $= 196$

211
- 15

196

4 Ans: $n(A_1 \cap A_2 \cap A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3) - n(A_1 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$
 $= 166 + 100 + 45 - 33 - 9 - 15 + 3$
 $= 314 - 57$
 $= 257$

5 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_1)$
 $= 257 - 166$
 $= 91$

6 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_3)$

$$= 257 - 45$$

$$= 212$$

(23)

7 Ans: $= n(A_1 \cup A_2 \cup A_3) - n(A_2)$

$$= 257 - 100$$

$$= 157$$

8 Ans:

$$n(\overline{A_1 \cup A_2 \cup A_3}) = \text{total no. of Integers} - n(A_1 \cup A_2 \cup A_3)$$

$$= 500 - 257$$

$$= 243$$