

Proposition :- Propositions (or) a statement is a declarative sentence which is either "TRUE" (or) "FALSE" but not both.

→ The truth or falsity of a proposition is called its truth-value.

→ these two values 'true' & 'false' are denoted by the symbols T & F.

respectively.

→ It can also denoted as,

True-1, False-0

Example:-

- ① Delhi is the capital of India
- ② Kolkata is a country
- ③ $2+3=4$

} Statements

∴ the above examples are ~~have~~ of ~~false~~ propositions (or) statements. either it is true (or) false.

Example: (2)

- ① How beautiful are you?
 - ② $x+y=3$
 - ③ Take one book.
- } not Statement

these are not propositions as they are not declarative in nature, i.e., they do not declare a definite truth value T or F.

⇒ Propositional Calculus:-

It is also known as Statement Calculus. It is a branch of mathematics that is used to describe a logical system or structure. A logical system consists of:

- ① A Universe of propositions.
- ② Truth tables (Axioms) for the logical operator.
- ③ Definition that explain equivalence & implication of proposition.

→ LOGIC CONNECTIVITIES:-

We can construct complicated statements from simple statements by using certain connecting words.

(a) expressions like.

"AND", "NOR", "OR", "NOT", "NAND", "IF, THEN"

"IF AND ONLY IF", such words.

are called connectivities.

→ the statement that we consider initially is simple statement / Atomic statement /

Primary / Primitive Statement.

→ the statements obtained by the use of connectivities is called "Molecular Statement" (a) Compound statement.

Name	Ref	Meaning
1) Negation	$\neg p$ or $\sim p$	not p
2) Conjunction	$p \wedge q$	p and q
3) Disjunction	$p \vee q$	p or q
4) Exclusive / implication	$p \rightarrow q$ or $p \Rightarrow q$	either p, q, not both
5) Biconditional	$p \leftrightarrow q$	p if and only if q

CONNECTIVES:

the word / phrases / symbols which are used to make a proposition by two or more propositions are called logical connectives or simply connectives.

→ there are five basic connectives. they are.

- ① Negation
- ② Conjunction
- ③ Disjunction
- ④ Conditional
- ⑤ Bi-conditional

1) NEGATION:

A statement obtained by inserting the word 'NOT' at an appropriate place in the given statement.

→ If p denotes a statement then the negation of p is written as $\neg p$ and read as 'not p '; ~~exp~~

→ If the truth value of p is T , then the truth value of p is F . Also if the truth value of p is F then the truth value of p is T .

Truth table for negation

p	$\sim p$
T	F
F	T

→ Negation is denoted as ' $\sim$ '.

2) CONJUNCTION

A compound statement is obtained by combining 2 given statements by inserting a word 'AND' in b/w them is called conjunction of the given statement.

→ The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T . Otherwise it

Truth table for conjunction

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

→ Conjunction is denoted as ' $\wedge$ '

3) DISJUNCTION

this statement is obtained by inserting a word 'OR' in between two statements.

Rule:-

→ the statement $p \vee q$ has the truth value F, only when both p and q have the truth value F. Otherwise it has truth value T.

TRUTH TABLE FOR DISJUNCTION

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

→ Disjunction is denoted as ' $\vee$ ',
read as ~~P OR Q~~ "P OR Q".

4) CONDITIONAL:-

If p and q are any two statements, then the statement $p \rightarrow q$ which is read as, 'If p; then q' is called a conditional statement (or) Implication and the connective is the conditional connective.

Rule: the statement $p \rightarrow q$ has the truth value F only when p is T and q is F & rest of the cases T.

Truth Table for Conditional

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

→ Conditional is denoted as ' $\rightarrow$ ',
read as 'p implies q'.

5) BI-CONDITIONAL :-

If, p and q are any two statements, then the statement $p \leftrightarrow q$ which is read as 'p if & only if q' and abbreviated as 'p iff q' is called as biconditional statement & the connective is the biconditional connective.

Rule: the statement $p \leftrightarrow q$ is true when p & q are same that is both are false or both are true.

Truth table for Bi-conditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

→ Bi-conditional is denoted as ' $\leftrightarrow$ '.
read as 'p if and only if q'.

Rule: the statement $p \leftrightarrow q$ is true when p & q are same that is both are false or both are true.

Truth table for Bi-conditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

→ Bi-conditional is denoted as ' $\leftrightarrow$ '.
read as 'p if and only if q'.

EXAMPLES FOR CONNECTIVES.

1) Negation:

i) p : Kolkata is a city.
 $\sim p$: Kolkata is not a city.

ii) p : 3 is prime number.
 $\sim p$: 3 is not a prime number.

2) Conjunction:

i) p : It is raining today.
 q : There are 10 chairs in the room.
 $p \wedge q$: It is raining today and there are 10 chairs in the room.

ii) p : $\sqrt{2}$ is irrational number.
 q : 9 is multiple of 3.
 $p \wedge q$: $\sqrt{2}$ is irrational number and 9 is multiple of 3.

⇒ Translate the following statement:

① Jack and Jill went up the hill into symbolic form using conjunction

Sol: Let p : Jack went up the hill.
 q : Jill went up the hill. $\rightarrow p \wedge q$
 $p \wedge q$: Jack and Jill went up the hill.

3) Disjunction

i) p : I shall go to the game.

ii) p : I shall go to the game.

q : I shall watch the game on television.
 $p \vee q$: I shall go to the game or watch the game on television.

ii) p : All triangles are equilateral.

q : $2+5=7$.

$p \vee q$: All triangles are equilateral or $2+5=7$.

4) Conditional:-

i) p : 2 is prime number.

q : 3 is prime number.

$p \rightarrow q$: if 2 is prime number then 3 is prime number.

⇒ Translate the following statement into symbolic form using conditional connective.

① 'The crop will be destroyed if there is a flood'

Sol: let p : the crop will be destroyed.
 q : there is a flood.

$p \rightarrow q$.

$\Rightarrow$ let us rewrite the given statement as
 If there is a flood, then the crop will be destroyed. So, the symbolic form of the given statement is $q \rightarrow p$.

Let p & q denote the statements.
 (i) p : You drive over 70 km per hour.
 q : You get a speeding ticket.

Write the following statements into symbolic form.

i) You will get a speeding ticket if you drive over 70 km per hour.

Sol: $p \rightarrow q$.

ii) Whenever you get a speeding ticket, you drive over 70 km per hour.

Sol: $q \rightarrow p$.

(ii) p : It rains
 q : the crop will grow.

$p \rightarrow q$: If it rains then the crop will grow.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

6) NAND ($\uparrow$):

NAND operation is combination of both NOT + AND.

→ It is denoted by $\uparrow$

→ $P \uparrow Q$ is false when two statements are true.

truth table for NAND

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

7) NOR ($\downarrow$):

NOR is combination of NOT + OR

→ It is denoted by $\downarrow$

→ $P \downarrow Q$ is true when two statements are false. the truth table follows.

truth table for NOR

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

8) Exclusive OR ($\oplus$):

Let P & Q are preposition, the exclusive OR of P & Q is denoted by $P \oplus Q$. is the preposition that is true when exactly one of P & Q is true.

truth table for Exclusive OR

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Example problems

1) Prove that $p \wedge (p \vee q)$

Sol:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

2) Find the truth table for $p \oplus q$

i) $p \wedge (\sim p \vee q)$

ii) $\sim p \wedge (\sim p \vee \sim q)$

Sol:

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

ii) $\sim p \wedge (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim p \wedge (\sim p \vee \sim q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	T

3) $p \wedge (p \rightarrow q)$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

4) $(p \wedge q) \Leftrightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \Leftrightarrow p \vee q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	F	T

Some more example problems:

- 1) $(p \vee q) \vee \sim p$
- 2) $(p \wedge (p \rightarrow q)) \rightarrow q$
- 3) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
- 4) $(q \wedge (p \rightarrow q)) \rightarrow p$
- 5) $\sim (p \rightarrow q) \rightarrow \sim p$
- 6) $(p \vee q) \wedge (p \rightarrow q)$
- 7) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
- 8) $(p \wedge (\sim q))$
- 9) $p \vee (\sim q)$
- 10) $p \rightarrow (q \vee r)$

Propositional Equivalence

Propositional equivalence is of 4 types.
they are:

- 1) Tautology.
- 2) Contradiction.
- 3) Contingency.
- 4) Logical Equivalence.

1) Tautology:-

A compound proposition which is true for all possible truth values of its components is called tautology.
Denoted by T_0 .

Ex: P.T. $[(p \rightarrow q) \wedge p] \rightarrow q$

P	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

As we can see every value of $[(p \rightarrow q) \wedge p] \rightarrow q$ is true. It is tautology.

2) Contradiction:

A compound proposition, which is 'false' for all possible truth values of its components, is called Contradiction.

→ Denoted by F_0 .

Ex: $P \vee T$ $(p \vee q) \wedge [(\sim p) \wedge (\sim q)]$

P	q	$p \vee q$	$\sim p$	$\sim q$	$(p \vee q) \wedge [(\sim p) \wedge (\sim q)]$
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	F

As we can say every value of $(p \vee q) \wedge [(\sim p) \wedge (\sim q)]$ is false. It is Contradiction.

3) Contingency:

A statement formula which is neither a tautology nor a contradiction is known as a Contingency.

Ex: Prove $(p \vee q) \wedge (\sim p)$

P	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

4) Logical Equivalence:

Two statements / propositions are said to be logically equivalent whenever P, Q have same truth values.

→ Represent as " $P \equiv Q$ ", " $P \leftrightarrow Q$ ", " $P \Leftrightarrow Q$ ".

→ Let 2 statements x & y are logically equivalent if any of the following 2 condition satisfy.

1) The truth table of each statement have the same truth values.

2) the Biconditional statement x if & only if y is a tautology. ($x \Leftrightarrow y$)

Ex:- $(p \rightarrow q) \Leftrightarrow (\sim p) \vee q$

p	q	$p \rightarrow q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As we can see it has same truth values.

→ $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

2) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

3) $(p \rightarrow q) \wedge (\sim p \rightarrow \sim q)$

4) $(p \vee q) \rightarrow (p \wedge q)$

→ Equivalence Formulas:-

Laws of Logic:-

1) Negation:-
Law of double negation for any statement P
 $\sim(\sim p) \Leftrightarrow p$

2) Idempotent law:- for any proposition p .
 $(p \vee p) \Leftrightarrow p$
 $(p \wedge p) \Leftrightarrow p$

3) Identity law:-
 $(p \vee F_0) \Leftrightarrow p$
 $(p \wedge T_0) \Leftrightarrow p$

4) Inverse law:-
 $(p \wedge \sim p) \Leftrightarrow F_0$
 $(p \vee \sim p) \Leftrightarrow T_0$

5) Domination law:-
 $(p \vee T_0) \Leftrightarrow T_0$
 $(p \wedge F_0) \Leftrightarrow F_0$

6) Commutative laws:- for statements p, q
 $(p \vee q) \Leftrightarrow (q \vee p)$
 $(p \wedge q) \Leftrightarrow (q \wedge p)$

7) Absorption law:-

$$(p \vee (p \wedge q)) \Leftrightarrow p$$

$$(p \wedge (p \vee q)) \Leftrightarrow p$$

8) De Morgan's law:-

$$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

9) Associative law:- for three statements p, q, r.

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

10) Distributive law:-

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

11) Conditional:-

$$\sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$$

$$(p \rightarrow q) \Leftrightarrow \sim p \vee q$$

4) Table for negation of compound statements

<u>Statement</u>	<u>Negation</u>
$\sim p$	p
$p \wedge q$	$\sim p \vee \sim q$
$p \vee q$	$\sim p \wedge \sim q$
$p \rightarrow q$	$p \wedge \sim q$

* logical equivalence involving conditional statement. (Implication or If-then)

	p	q	$p \rightarrow q$	$\sim p \vee q$
1) $p \rightarrow q \equiv (\sim p \vee q)$	T	T	T	T
2) $p \rightarrow q \equiv \sim q \rightarrow \sim p$	T	F	F	F
3) $p \vee q \equiv \sim p \rightarrow q$	T	T	T	T
4) $p \wedge q \equiv \sim (q \rightarrow \sim p)$	T	F	F	F
5) $\sim (p \rightarrow q) \equiv p \wedge \sim q$	T	T	T	T
6) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$	T	T	T	T
7) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$	T	T	T	T
8) $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$	T	T	T	T
9) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	T	T	T	T

* logical equivalence Involving Bi-Conditional Statement

$$1) p \Leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$2) p \Leftrightarrow q \equiv r(p \Leftrightarrow r) \vee q$$

$$3) p \Leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$4) r(p \Leftrightarrow q) \equiv p \Leftrightarrow r q$$

Example problems.

1) "If x is not a real number, then it is not a rational number & not an irrational number."

Let $p: x$ is real number.
 $q: x$ is rational number.
 $r: x$ is irrational number.

$$[\neg p \rightarrow (\neg q \wedge \neg r)]$$

$$\begin{aligned} \neg[\neg p \rightarrow (\neg q \wedge \neg r)] &\equiv \neg p \wedge \neg(\neg q \wedge \neg r) \\ &\equiv \neg p \wedge (\neg \neg q \vee \neg \neg r) \\ &\equiv \neg p \wedge (q \vee r) \end{aligned}$$

$\therefore$ negation of given proposition is

" x is not a real number and it is a rational number & an irrational number."

Example:-

1) Show that $r(p \vee q)$ and $r p \wedge r q$ are logically equivalent.

P	q	r p	r q	p v q	r(p v q)	r p ^ r q
T	T	F	F	T	F	F
T	F	F	F	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

therefore $r(p \vee q) \equiv r p \wedge r q$

2) Prove that $(p \vee q) \wedge [\neg(\neg p) \wedge q]$ is logically equivalent using law of logic.

$$(p \vee q) \wedge [\neg(\neg p) \wedge q]$$

$$(p \vee q) \wedge [(\neg \neg p) \wedge q] \text{ De Morgan's law}$$

$$(p \vee q) \wedge [p \vee \neg q] \text{ Double negation law}$$

$$p \vee (q \wedge \neg q) \text{ distributive law}$$

$$p \vee F \text{ De Morgan's law}$$

$$p \text{ Identity law}$$



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3) Prove that $(\neg p \wedge \neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$$\begin{aligned} & (\neg p \wedge \neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \\ \Rightarrow & ((\neg p \wedge \neg q) \wedge r) \vee ((q \vee p) \wedge r) \\ & \text{(Associative \& distributive law)} \end{aligned}$$

$$\Rightarrow (\neg(p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$$

[De Morgan's law]

$$\Rightarrow (\neg(p \vee q) \vee (p \vee q)) \wedge r$$

[distributive law]

$$\Rightarrow T \wedge R \quad [\because \neg p \vee p \Leftrightarrow T]$$

$$\rightarrow R //$$

4) Show $((p \vee q) \wedge \neg(p \wedge (q \vee r))) \vee ((p \wedge q) \vee (p \wedge r))$ is a tautology

Sol: By De Morgan's law, we have.

$$\neg p \wedge \neg q \Leftrightarrow \neg(p \vee q)$$

$$\neg p \vee \neg r \Leftrightarrow \neg(p \wedge r)$$

therefore,

$$(p \wedge r) \vee (\neg p \wedge \neg r) \Leftrightarrow (p \vee \neg p) \wedge (r \vee \neg r)$$

$$\Rightarrow T \wedge T$$

$$\Rightarrow T$$

also,

$$\neg(p \wedge (q \vee r))$$

$$\Rightarrow \neg(p \wedge \neg(q \wedge r))$$

$$\Rightarrow p \vee (q \wedge r)$$

$$\Rightarrow (p \vee q) \wedge (p \vee r)$$

Hence,

$$((p \vee q) \wedge \neg(p \wedge (q \vee r))) \vee ((p \wedge q) \vee (p \wedge r))$$

$$\Rightarrow (p \vee q) \wedge (p \vee r) \vee (p \wedge q) \vee (p \wedge r)$$

$$\Rightarrow (p \vee q) \wedge (p \vee r)$$

thus,

$$((p \vee q) \wedge \neg(p \wedge (q \vee r))) \vee ((p \wedge q) \vee (p \wedge r))$$

$$\Rightarrow [(p \vee q) \wedge (p \vee r)] \vee [(p \wedge q) \vee (p \wedge r)]$$

$$\Rightarrow T$$

Hence the given formula is a tautology

5) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$(p \wedge q) \rightarrow (p \vee q)$$

$$\Rightarrow \neg(p \wedge q) \vee (p \vee q) \quad (\because p \rightarrow q \Leftrightarrow \neg p \vee q)$$

$$\Rightarrow (\neg p \vee \neg q) \vee (p \vee q) \quad (\text{by De Morgan's law})$$

$$\Rightarrow (\neg p \vee p) \vee (\neg q \vee q) \quad (\text{by Associative law \& Commutative law})$$

$$\Rightarrow (T \vee T) \quad (\text{by negation laws})$$

$$\Rightarrow T.$$

Hence, the result.

6) Prove that $\sim[\sim((p \vee q) \wedge r) \vee \sim q]$

$$\sim(\sim((p \vee q) \wedge r) \vee \sim q)$$

$$\Rightarrow \sim[\sim((p \vee q) \wedge r) \wedge \sim(\sim q)]$$

$$\Rightarrow \sim[\sim((p \vee q) \wedge r) \wedge q] \quad (\because \text{De Morgan's law})$$

$$\Rightarrow ((p \vee q) \wedge r) \wedge q \quad (\because \text{double negation law})$$

$$\Rightarrow ((p \vee q) \wedge q) \wedge r \quad (\text{using associative, commutative laws})$$

$$\Rightarrow ((p \vee q) \wedge q) \wedge r \quad (\text{associative law})$$

$$\Rightarrow q \wedge r \quad (\text{using absorption law})$$

$$\Rightarrow [p \vee (p \wedge (p \vee q))] \Leftrightarrow p$$

$$p \vee p \quad (\text{Absorption law})$$

$$p \quad (\text{Idempotent law})$$

$$(p \vee F) \wedge (q \vee T) \Leftrightarrow p$$

$$p \wedge (q \vee T) \quad (\text{Identity law})$$

$$p \wedge T \quad (\text{domination law})$$

$$p \equiv p \quad (\text{Identity law})$$

$$\sim(\sim p) \vee [(p \vee F) \wedge \sim(\sim q)] \Leftrightarrow p$$

$$p \vee [(p \vee F) \wedge q] \quad (\text{double negation law})$$

$$p \vee [p \wedge q] \quad (\text{Identity law})$$

$$p \quad (\text{Absorption law})$$

$$10) [\sim(p \wedge q) \wedge q] \Leftrightarrow \sim(p \vee q)$$

$$11) p \cdot T(p \vee q) \wedge \sim(\sim p \vee q)$$

$$12) \sim(\sim((p \vee q) \wedge r) \vee \sim q)$$

$$13) ((p \rightarrow q) \rightarrow r) \Leftrightarrow (\neg p \vee q) \rightarrow r$$

$$14) p \vee (p \wedge (p \vee r)) \Leftrightarrow p$$

$$15) [(p \rightarrow q) \wedge (\sim q \wedge (q \vee \sim q))] \Leftrightarrow \sim(q \vee p)$$

$$\text{Sol: } [(p \rightarrow q) \wedge (\sim q \wedge (\sim q \vee q))] \text{ (commutative law)}$$

$$\Rightarrow (p \rightarrow q) \wedge \sim q \quad [\because \text{Absorption law}]$$

$$\Rightarrow (\sim [(p \rightarrow q) \rightarrow q]) \quad [\because \sim(p \rightarrow q) \Leftrightarrow (p \wedge \sim q)]$$

$$\Rightarrow \sim(\sim(p \rightarrow q) \wedge q) \quad [\because (p \rightarrow q) \Leftrightarrow (\sim p \vee q)]$$

$$\Rightarrow \sim[(p \wedge \sim q) \vee q]$$

$$\Rightarrow \sim[q \vee (p \wedge \sim q)] \quad [\because \text{Commutative law}]$$

$$\Rightarrow \sim[(q \vee p) \wedge (q \vee \sim q)] \quad [\text{Distributive law}]$$

$$\Rightarrow \sim((q \vee p) \wedge T) \quad [q \wedge \sim q \text{ is tautology}]$$

$$\Rightarrow \sim(q \vee p) \quad [\text{Identity law}]$$

Converse, Inverse & Contrapositive.

IMPLICATION

Implication or if-then is also called a conditional statement i.e. $p \rightarrow q$. It has two parts.

$$p \rightarrow q \text{ is } \sim p \rightarrow \sim q$$

i) Hypothesis (p)

ii) Conclusion (q)

An Inverse of the conditional statement is the negation of both the hypothesis & the conclusion.

If the statement is "If p, then q", the Inverse will be "If not p, then not q". ($\sim p \rightarrow \sim q$)

Ex: p: If you do your home work
q: you will not be punished.

Inverse: p: If you do not do your home work.

q: then you will be punished.

Ex: p: Today is Sunday
q: I will go for a walk.

Inverse p: If today is not Sunday.

q: Then I will not go for a walk.

2) Converse:

the converse of the conditional statement is computed by interchanging the hypothesis and the conclusion.

→ If the statement is "If p , then q " then converse will be "If q , then p ".
($p \rightarrow q$)
($q \rightarrow p$)

→ the converse of $p \rightarrow q$ is $q \rightarrow p$.

① Ex: If you do your homework, you will not be punished.

Converse: If you will not be punished, then you will do your homework.

② Ex: Today is Sunday. I will go for a walk.

Converse: If I am going for a walk, then it is Sunday.

3) Contrapositive

the contrapositive of the conditional statement is computed by interchanging the hypothesis and conclusion of Inverse Statement.

→ If the statement is "If p , then q " the Contrapositive "If not q , then not p ".

→ the contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

Ex: If you do your homework, you will not be punished.

Contrapositive: If you are punished, then you did not do your homework.

② Ex: Today is Sunday. I will go for a walk.

Contrapositive: If I will not go for a walk, then today is not Sunday.

($p \rightarrow q$) $\Leftrightarrow$ ($\sim q \rightarrow \sim p$)

Converse, Inverse, Contrapositive.

Consider a $p \rightarrow q$, then

- 1) $q \rightarrow p$ is called converse of $p \rightarrow q$.
- 2) $\sim p \rightarrow \sim q$ is called inverse of $p \rightarrow q$.
- 3) $\sim q \rightarrow \sim p$ is called contrapositive of $p \rightarrow q$.

P	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$$\therefore p \rightarrow q \Leftrightarrow (\sim q \rightarrow \sim p).$$

$$\therefore q \rightarrow p \Leftrightarrow (\sim p \rightarrow \sim q).$$

Exercise

- 1) State the converse, inverse & contrapositive of the following conditionals.
- 2) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Inverse: If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

Contrapositive: If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

- 3) If a real number x^2 is greater than zero, then x is not equal to zero.

Converse: If a real number x is not equal to zero, then x^2 is greater than zero.

Inverse: If a real number x^2 is not greater than zero, then x is equal to zero.

Contrapositive: If a real number x is equal to zero, then x^2 is not greater than zero.

iii) If a triangle is not isosceles, then it is not equilateral.

Converse: If a triangle is not equilateral, then it is not isosceles.

Inverse: If a triangle is isosceles, then it is equilateral.

Contrapositive: If a triangle is equilateral, then it is isosceles.

2) Write down the conditionals given in the previous exercise by using

- a) the 'necessary condition' language
- b) the 'sufficient condition' language

1) A necessary condition for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

2) A necessary condition for real number n^2 to be greater than zero is that

n is not equal to zero.

3) For a triangle to be non-isosceles it is necessary that it is not equilateral.

⇒ Normal form:-

Conjunction → product → AND → $\wedge$

Disjunction → Sum → OR → $\vee$

⇒ It means of representing propositional expression.

⇒ Here, we use the words (product, Sum)

↳ Elementary product:-

We call it as fundamental conjunction.

⇒ A product of variables & their negation in a formula is called Elementary product.

$$P \wedge Q$$

$$\sim P \wedge Q$$

$$P \wedge \sim Q$$

$$\sim P \wedge \sim Q$$

↳ Elementary Sum:- Fundamental disjunction

⇒ Sum of variables & their negation is called elementary sum.

$$P \vee Q$$

$$\sim P \vee Q$$

$$P \vee \sim Q$$

$$\sim P \vee \sim Q$$

⇒ There are two types of normal form.

They are:

① Disjunction Normal form (DNF)

② Conjunction Normal form (CNF)

1) DNF:-

A formula which is equivalent to given formula & which consists of "Sum of elementary products".

$$(P \wedge Q) \vee (\sim P \wedge \sim Q)$$

Product Sum Product

2) CNF:-

A formula which is equivalent to given formula & which consists of "Product of sum terms" is called CNF.

$$(P \vee Q) \wedge (\sim P \vee \sim Q)$$

Procedure:

- 1) Replace implication, Bi-implication by an equivalent expression containing $\wedge, \vee, \neg$ only.

Ex: $P \rightarrow Q \equiv (\neg P \vee Q)$

$$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$$

- 2) Eliminate negation before Sum & product by using double negation or De Morgan's law.

- 3) Apply distributive law until DNF or CNF is obtained.

* Example problem for DNF :-

- 1) Find DNF of $P \wedge (P \rightarrow Q)$.

Sol: Given,

$$P \wedge (P \rightarrow Q)$$

$$P \wedge (\neg P \vee Q) \quad [\because \text{conditional law}]$$

$$(P \wedge \neg P) \vee (P \wedge Q) \quad [\because \text{distributive law}]$$

$\therefore$ It is in DNF form.

$$\boxed{(P \wedge Q) \vee (\neg P \wedge \neg Q)}$$

Q) $\neg [P \rightarrow (Q \wedge R)]$

Sol: Given,

$$\neg [P \rightarrow (Q \wedge R)]$$

$$P \rightarrow Q \equiv \neg P \vee Q$$

$$\Rightarrow \neg [(\neg P) \vee (Q \wedge R)] \quad [\because \text{conditional law}]$$

$$\Rightarrow P \wedge \neg (Q \wedge R) \quad [\because \text{double negation law}]$$

$$\Rightarrow P \wedge (\neg Q \vee \neg R) \quad [\because \text{De Morgan's law}]$$

$$\Rightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \quad [\because \text{distributive law}]$$

$\therefore$ It is in DNF form.

* Example problem for CNF :-

- 1) Find out CNF of $\neg [P \wedge (P \rightarrow Q)]$.

Sol: Given,

$$\neg [P \wedge (P \rightarrow Q)]$$

$$\Rightarrow \neg P \vee \neg (P \rightarrow Q) \quad [\because \text{De Morgan's law}]$$

$$\Rightarrow \neg P \vee \neg ((\neg P) \vee Q) \quad [\because \text{conditional law}]$$

$$\Rightarrow \neg P \vee (P \wedge \neg Q) \quad [\because \text{De Morgan's law}]$$

$$\Rightarrow (\neg P \vee P) \wedge (\neg P \vee \neg Q) \quad [\because \text{distributive law}]$$

$\therefore$ It is in CNF form.

$$\boxed{(\neg P \vee Q) \wedge (\neg P \vee \neg Q)}$$

$$2) \sim (I \rightarrow \sim Q) \wedge R$$

$$\text{Sol: } \sim [I \rightarrow \sim Q] \vee \sim (\sim R) \quad [\text{De Morgan's law}]$$

$$\Rightarrow \sim [(\sim I) \vee \sim Q] \vee R \quad [\text{Conditional law}]$$

$$\Rightarrow (\sim \sim I \wedge Q) \vee R \quad [I]$$

$$\Rightarrow \sim (\sim I \rightarrow \sim Q) \vee R$$

$$\Rightarrow (\sim R \wedge \sim (\sim Q)) \vee R$$

$$\Rightarrow (\sim P \vee R) \wedge (Q \vee R) \quad [\text{Distributive law}]$$

$\therefore$ It is in CNF form.

Principle Normal form

Given two simple propositions p & q .

The compound propositions are of two

types: ① Minterms & ② Max terms.

* Minterms:-

For 2 propositions for 3 propositions
(p, q) ($2^2 = 2 = 4$) (p, q, r) ($2^3 = 8$)

$$p \wedge q$$

$$\sim p \wedge q$$

$$p \wedge \sim q$$

$$\sim p \wedge \sim q$$

$$p \wedge q \wedge r$$

$$\sim p \wedge q \wedge r$$

$$\sim p \wedge \sim q \wedge r$$

$$p \wedge q \wedge \sim r$$

$$\sim p \wedge q \wedge \sim r$$

$$p \wedge \sim q \wedge \sim r$$

$$\sim p \wedge \sim q \wedge \sim r$$

$\rightarrow$ these are the terms involving p & q .

* Max terms:-

the duals of these minterms.

for 2 propositions

$$(p, q)$$

$$p \vee q$$

$$\sim p \vee q$$

$$p \vee \sim q$$

$$\sim p \vee \sim q$$

for 3 propositions

$$(p, q, r)$$

$$p \vee q \vee r$$

$$\sim p \vee q \vee r$$

$$\sim p \vee \sim q \vee r$$

$$p \vee q \vee \sim r$$

$$\sim p \vee q \vee \sim r$$

$$p \vee \sim q \vee \sim r$$

$$\sim p \vee \sim q \vee \sim r$$



① Principle normal form consists of 2 types they are:

- ① Principle Disjunctive normal form (PDNF)
- ② Principle Conjunctive normal form (PCNF)

→ The DNF & CNF of statement formula is not unique in order to obtain a unique result of a given statement formula we introduce principle normal form.

1) Principle Disjunctive Normal form (PDNF):

A Compound proposition involving two simple propositions p and q , an equivalent compound proposition consisting of disjunction of the minterms involving p and q only is known as its principle disjunctive normal form (or sum-of-products canonical form).

Minterms:

For a given no. of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

P	Q	$P \wedge Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

→ From the above truth tables of these minterms of P & Q , it is clear that:

- i) no two minterms are equivalent.
- ii) Each minterm has the truth value T for exactly one combination of the truth values of the variables P & Q .

* Obtain PDNF of a given formula by truth table

- i) Construct a truth table of the given formula.
- ii) for every truth value that has the value T in the truth table of the given formula, select the miniterms which also has the value T. For the same combination of the truth values of P and Q , there is only one miniterm.
- iii) the disjunction of all miniterms will then be equivalent to the given formula.

⇒ The PDNF of $P \rightarrow Q$ is $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$
 $\Rightarrow P \rightarrow Q = (P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

Example: Obtain the PDNF of $P \rightarrow Q$.
 Solution: from the truth table of $P \rightarrow Q$.

P	Q	$P \rightarrow Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \sim Q$
F	T	T	$\sim P \wedge Q$
F	F	T	$\sim P \wedge \sim Q$

→ The PDNF of $P \rightarrow Q$ is $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$
 $\therefore P \rightarrow Q \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

Example: Obtain the PDNF for $(P \wedge Q) \vee (\sim P \wedge Q) \vee (Q \wedge R) \Rightarrow S$

Sol:

P	Q	R	Minterm	$P \wedge Q$	$\sim P \wedge R$	$Q \wedge R$	S
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \sim R$	T	F	F	T
T	F	T	$P \wedge \sim Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \sim Q \wedge \sim R$	F	F	F	F
F	T	T	$\sim P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\sim P \wedge Q \wedge \sim R$	F	F	F	F
F	F	T	$\sim P \wedge \sim Q \wedge R$	F	T	F	T
F	F	F	$\sim P \wedge \sim Q \wedge \sim R$	F	F	F	F

$\therefore$ the PDNF of $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$ is $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge \sim R)$

⑥ Without constructing the truth table.

In order to obtain the PDNF of a given formula is constructed as follows:

- 1) First replace $\rightarrow$, by their equivalent formula containing only $\wedge$, $\vee$ and $\sim$.
- 2) Next, negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.
- 3) Any elementary product which is a contradiction is dropped.
- $\rightarrow$ Minterms are obtained in the disjunction by introducing the missing factors.
- $\rightarrow$ Identical minterms appearing in the disjunctions are deleted.

Example: Obtain the PDNF of $P \rightarrow Q$

① $\sim P \vee Q$

② $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

③ $P \rightarrow Q \Leftrightarrow \sim P \vee Q \Leftrightarrow (\sim P \wedge T) \vee (Q \wedge T) \quad [\because P \wedge T \Leftrightarrow P]$

$\Rightarrow (\sim P \wedge (Q \vee \sim Q)) \vee (Q \wedge (P \vee \sim P))$

$[\because P \vee \sim P \Leftrightarrow T]$

$\Rightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (Q \wedge P) \vee (Q \wedge \sim P)$

$[\because P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)]$

$\Rightarrow (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q) \quad [\because P \vee \sim P \Leftrightarrow T]$

④ $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

$\Rightarrow (P \wedge Q \wedge T) \vee (\sim P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$

$\Rightarrow (P \wedge Q \wedge (R \vee \sim R)) \vee (\sim P \wedge R \wedge (Q \vee \sim Q)) \vee (Q \wedge R \wedge (P \vee \sim P))$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P)$

$\Rightarrow P \vee (P \wedge Q) \Leftrightarrow P$

$\Rightarrow P \vee (\sim P \wedge Q) \Leftrightarrow P \vee Q$



2) Principal Conjunctive Normal Form

(PCNF) Max terms

The dual of a minterm is called a maxterm. For a given number of variables, the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

→ Each of the maxterms has the truth value 0 for exactly one combination of the truth values of the variables.

2) Principal Disjunctive Normal Form

(PDNF)

A compound proposition involving two simple propositions p and q , an equivalent compound proposition $\vee$ consisting of conjunctions of the maxterms involving p and q only is known as its

Principal Conjunctive normal form (product - of - sums Canonical form)

→ the method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

→ Example: Obtain the PCNF of the formula $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

$$\text{Sol: } (\sim p \rightarrow r) \wedge (q \leftrightarrow p)$$

$$\Rightarrow (\sim(\sim p) \vee r) \wedge ((q \rightarrow p) \wedge (p \rightarrow q))$$

$$\Rightarrow (p \vee r) \wedge ((\sim q \vee p) \wedge (\sim p \vee q))$$

$$\Rightarrow (p \vee r \vee f) \wedge ((\sim q \vee p \vee f) \wedge (\sim p \vee q \vee f))$$

$$\Rightarrow [(p \vee r) \vee (q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (p \wedge \sim p)]$$

$$\wedge [(\sim p \vee q) \vee (p \wedge \sim p)]$$

$$\Rightarrow (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (p \vee \sim q \vee r)$$

$$\wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$$

$$\Rightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r)$$

$$\wedge (\sim p \vee q \vee \sim r)$$

∴ It is required PCNF.



Example Obtain the prime PCNF of the following.

i) $(\neg p \rightarrow q) \wedge (q \leftrightarrow p)$

ii) $(p \wedge q) \vee (\neg p \wedge q)$

Sol: i) $(\neg p \rightarrow q) \wedge (q \leftrightarrow p)$

$(\neg p \rightarrow q) \wedge (q \leftrightarrow p)$

$\Rightarrow (p \vee q) \wedge \{ (q \rightarrow p) \wedge (p \rightarrow q) \}$

$\Rightarrow (p \vee q) \wedge \{ (\neg q \vee p) \wedge (\neg p \vee q) \}$

$\Rightarrow (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q)$

ii) $(p \wedge q) \vee (\neg p \wedge q)$

$\Rightarrow \{ (p \wedge q) \vee \neg p \} \wedge \{ (p \wedge q) \vee q \}$

$\Rightarrow (p \vee \neg p) \wedge (q \vee \neg p) \wedge \{ (p \vee q) \wedge (q \vee \neg p) \}$

$\Rightarrow T_0 \wedge (q \vee \neg p) \wedge \{ (p \vee q) \wedge T_0 \}$

$\Rightarrow (q \vee \neg p) \wedge (p \vee q)$

Obtain PCNF

i) $p \vee q$

Sol: $(p \wedge T_0) \vee (q \wedge T_0)$

$(p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$

$\Rightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$

$\Rightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

PCNF using truth tables.

for every truth value 'p' of a given formula select the maxterms which also has the truth value 'p' for some combination of negation of truth value of the statement formula.

Sol: $p \leftrightarrow q$			
P	q	$p \leftrightarrow q$	maxterms
T	T	T	$\neg p \vee \neg q$
T	F	F	$\neg p \vee q$
F	T	F	$p \vee \neg q$
F	F	T	$p \vee q$

the equivalent PCNF of $p \leftrightarrow q$ is $(\neg p \vee q) \wedge (p \vee \neg q)$

→ Without using truth table:

1) Replace the conditional & bi-conditional connectives in the given statement by $\wedge, \vee, \sim$

2) apply negation to the variable by using demorgan followed by distributive laws.

3) Any elementary sum which is a tautology can be dropped.
($P \vee \sim P$) = T.

4) Max terms are obtained in conjunction by introducing missing factors.

5) Identical max terms appearing in the conjunction are deleted.

DNF

1) $p \rightarrow [(p \rightarrow q) \wedge \sim (\sim q \vee \sim p)]$

2) $\sim [p \rightarrow (q \wedge r)]$

3) $p \wedge (p \rightarrow q)$

CNF

1) $\sim (p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$

2) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

3) $\sim (p \rightarrow q) \vee (r \rightarrow p)$

PDNF

1) $\sim p \vee q$

2) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

3) $p \rightarrow [(p \rightarrow q) \wedge \sim (\sim q \vee \sim p)]$

PCNF

1) $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

2) $(\sim p \rightarrow q) \wedge (q \leftrightarrow p)$

3) $(p \wedge q) \vee (\sim p \wedge q)$

→ Well formed formula:

Statement formula:

A statement formula is an expression consisting of connectives, statements & parenthesis is known as well formed formula.

Rules for well formed formula:

1) A statement variable standing alone itself is known as well formed formula.

2) If a statement p is well formed then negation of p is well formed formula.

3) If p & q are well formed then $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ & $(p \leftrightarrow q)$ are well formed formulas.

Ex:- $\sim(p \vee q)$ is well formed.

p, q - variables

$\sim$ - connective

$()$ - parenthesis

→ Quantifiers:

Quantifiers is nothing, but collection of words.

All $\forall$ there exist $\exists$
None $\neg \forall$ some $\exists$

→ Theory of Inference for the Statement Calculus.

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as inference theory.

→ If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof & the argument is called a valid argument or conclusion is called a valid conclusion.

Rule of Inference:

Consider a set of propositions $P_1, P_2, P_3, \dots, P_n$ & a proposition C then a compound proposition of the form is $(P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n) \rightarrow C$ is called an argument.

→ Here $P_1, P_2, P_3, \dots, P_n$ are called premises of the argument & C is called conclusion of the argument. ∴ therefore the tabular form of the compound proposition is

P_1
P_2
P_3
$\vdots$
P_n
$\hline C$

Consistency & Inconsistency of premises:

1) Consistency:

A set of premises $H_1, H_2, H_3, \dots, H_m$ are said to be consistent if their conjunction $H_1 \wedge H_2 \wedge H_3 \wedge \dots \wedge H_m$ has the truth value 'T' in at least one possible situation.

Ex: If the following premises are consistent.

$$(p \vee q) \wedge (\sim p).$$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge (\sim p)$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

In the given statement is result there is at least one True value.

∴ It is a the above premises are consistent.



d) Inconsistency

A set of premises H_1, H_2, H_3 are said to be inconsistent if their conjunction $H_1 \wedge H_2 \wedge H_3$ has the truth value 'F' (false) in every possible situation is called Inconsistency.

Ex: If the following premises are inconsistent:

$$(\sim p \wedge q) \wedge (p)$$

p	q	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \wedge p$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

$\therefore$ the above premises are inconsistent because $(\sim p \wedge q) \wedge p$ is false in all possible situations.

Valid and Invalid Arguments

An argument is a set of initial statements are called premises followed by a conclusion.

→ An argument is valid if & only if in every case where all the premises are true otherwise the argument is invalid.

→ An argument with premises $P_1, P_2, P_3, \dots, P_n$ & conclusion C is said to be a valid if whenever each premises $P_1, P_2, P_3, \dots, P_n$ is true then the conclusion C is also true.

→ In other words the argument $(P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n) \rightarrow C$

→ the conclusion is true only in the case of valid argument.

Q: Is the following argument is valid or not.

Argument - If my computer is crashes then I will lose all my photos.

$\rightarrow$ I have not lost all my photos. therefore my computer has not crashed.

Sol:

P: my computer crashes

Q: I'll lose my photos.

$P \rightarrow Q$

$\neg Q$

$\neg P$

Conclusion: $\neg P$

truth table

$$((P \rightarrow Q) \wedge \neg Q) \rightarrow \neg P \Rightarrow R$$

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$(P \rightarrow Q) \wedge \neg Q$	R
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

$\therefore$ the given argument is valid.

Rules of Inference

Rules of Inference	Tautology	Name
1) $\frac{p \quad p \rightarrow q}{q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens (law of detachment)
2) $\frac{\sim q \quad p \rightarrow q}{\sim p}$	$[\sim q \wedge (p \rightarrow q)] \rightarrow \sim p$	Modus tollens (law of Contradiction)
3) $\frac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism (Transitive rule)
4) $\frac{p \vee q \quad \sim q}{q}$	$[(p \vee q) \wedge \sim q] \rightarrow q$	disjunctive syllogism
5) $\frac{p}{p \vee q}$	$p \rightarrow (p \vee q)$ $q \rightarrow (p \vee q)$	Addition

6) $\frac{p \wedge q}{p}$	$(p \wedge q) \rightarrow p$ $(p \wedge q) \rightarrow q$	Simplification
7) $\frac{p \quad q}{p \wedge q}$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	conjunction
8) $\frac{p \vee q \quad \sim p \vee r}{q \vee r}$	$[(p \vee q) \wedge (\sim p \vee r)] \rightarrow (q \vee r)$	Resolution

→ Theory of Inference for the Statement Calculus

the truth table technique becomes tedious if the premises contains a large no. of statement variables to overcome this limitation we follow other possible methods without using truth table. . .

there are 3 types of inference rules

- 1) Rule - P
 - 2) Rule - T
 - 3) Rule - CP
- ∴ Rule P & T are called the 2 basic rules of Inference.

1) Rule-P:

We may introduce a premise at any step in the derivation.

2) Rule-T:

A formula S may be introduced in the derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

3) Rule-CP:

If a formula S can be derived from another formula R & a set of premises, then the statement $R \rightarrow S$ can be derived from the set of premises alone.

* Implication Rules:

$I_1: p \wedge q \Rightarrow p$
 $I_2: p \wedge q \Rightarrow q$ } Simplification

$I_3: p \Rightarrow p \vee q$
 $I_4: q \Rightarrow p \vee q$ } Addition

$I_5: \sim p \Rightarrow p \rightarrow q$

$I_6: \sim q \Rightarrow p \rightarrow q$

$I_7: \sim(p \rightarrow q) \Rightarrow p$

$I_8: \sim(p \rightarrow q) \Rightarrow q$

$I_9: p, q \Rightarrow p \wedge q$

$I_{10}: \sim p, p \vee q \Rightarrow q$ (Disjunctive Syllogism)

$I_{11}: p, p \rightarrow q \Rightarrow q$ (Modus ponens)

$I_{12}: \sim q, p \rightarrow q \Rightarrow \sim p$ (Modus Tollens)

$I_{13}: p \rightarrow q, q \rightarrow r \Rightarrow p \rightarrow r$ (Hypothetical Syllogism)

$I_{14}: p \vee q, p \rightarrow r, q \rightarrow r \Rightarrow r$ (Dilemma)

* Equivalence Rules

$$E1: \sim \sim p \Rightarrow p \text{ (Double negation)}$$

$$E2: p \wedge q \Leftrightarrow q \wedge p \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Commutative}$$

$$E3: p \vee q \Leftrightarrow q \vee p$$

$$E4: (p \wedge q) \wedge r \Leftrightarrow (p) \wedge (q \wedge r) \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Associative}$$

$$E5: (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

$$E6: p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Distributive}$$

$$E7: p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$E8: \sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{De Morgan's}$$

$$E9: \sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$E10: p \vee p \Leftrightarrow p$$

$$E11: p \wedge p \Leftrightarrow p$$

$$E12: x \vee (p \wedge \sim p) \Leftrightarrow x$$

$$E13: x \wedge (p \vee \sim p) \Leftrightarrow x$$

$$E14: x \wedge (p \vee \sim p) \Leftrightarrow x$$

$$E15: x \wedge (p \wedge \sim p) \Leftrightarrow \perp$$

$$E16: p \rightarrow q \Leftrightarrow \sim p \vee q$$

$$E17: \sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$$

$$E18: p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$$

$$E19: p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$$

$$E20: \sim (p \rightarrow q) \Leftrightarrow p \wedge \sim q$$

$$E21: p \Leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$E22: p \Leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$$

Example problems

1) Demonstrate that R is valid inference from the given premises.

$$P, p \rightarrow q, q \rightarrow r.$$

Premises: p [Rule P]

$\{1\}$ p [Rule P]

$\{2,3\}$ $p \rightarrow q$ [Rule P]

$\{1,2,3\}$ q [Rule $\rightarrow$ $\{1,2\}$]

$\{4\}$ $q \rightarrow r$

$\{1,2,4\}$ r [Rule $\rightarrow$ $\{1,2,4\}$]

$\therefore$ the given premises is valid.

2. Show that RUS follows logically from the premises $CVD, (CVD) \rightarrow \sim H, \sim H \rightarrow (A \wedge \sim B)$ and $(A \wedge \sim B) \rightarrow RUS$

- Sol: {1} (1) CVD Rule P
 {2} (2) $CVD \rightarrow \sim H$ Rule P
 {1,2} (3) $\sim H$ Rule-T
 {4} (4) $\sim H \rightarrow (A \wedge \sim B)$ Rule P
 {1,2,4} (5) $A \wedge \sim B$ Rule-T
 {6} (6) $(A \wedge \sim B) \rightarrow RUS$ Rule P
 {1,2,4,6} (7) RUS Rule-T

$\therefore$ the given argument is valid.

3) If Sachin hits a century then he gets a free car.

Sol: let p - If Sachin hits a century.
 q - He gets a free car.

$$\frac{p \rightarrow q, p}{q}$$

$\therefore q$ [Modus Ponens].

4) If I study then I do not fail in the exam. If I do not fail in the exam then my father gifts a two wheeler to me.

Sol: let p - I study
 q - I do not fail in the exam
 r - my father gifts two wheeler to me.

$$\frac{p \rightarrow q, q \rightarrow r}{p \rightarrow r}$$

* Theory of Inference for predicate calculus.

Certain additional rules are required to deal with formulas involving in quantifiers -

- $\rightarrow$ the elimination of quantifiers can be done by using the rules of specifications called as VS and ES.
- $\rightarrow$ To Prefix the correct quantifiers we need the rules of generalization called UG & EG.

→ Rule.

* Rule US (Universal Specification).

If a statement of the form $\forall(x)[P(x)]$ is assumed to be true then the universal quantifier can be dropped to obtain $P(t)$ is true for an object t in the universe.

→ the symbolic form of this rule is

$$\forall(x)[P(x)]$$

$P(t)$ for all ' t '.

* Rule VG (Universal Generalisation).

If a statement $P(t)$ is true for each element t of the universe then the universal quantifier may be prefixed to obtain

$$\forall(x)[P(x)].$$

→ the symbolic form of this rule is

$$P(t) \text{ for all } 't'$$

$$\forall(x)[P(x)].$$

* Rule EG (Existential Generalization).

If a statement $P(t)$ is true for some element t in the universe then the existential quantifier may be prefixed to obtain $\exists(x)[P(x)]$ is true.

→ the symbolic form of this rule is

$$P(t) \text{ for some } 't'$$

$$\exists(x)[P(x)]$$

Example

1) Verify the validity of following argument.

① If every living thing is a plant or an animal.

② John's gold fish is alive and it is not a plant.

③ All animals have heart, therefore John's gold fish has a heart.

Let us consider.

x : Every living thing
 $P(x)$: x is a plant
 $A(x)$: x is an animal
 $H(x)$: x has heart
 $J(x)$: John's gold fish

Represent the given statements in symbolic form.

I) Every living thing is a plant or animal.

$$H(x) [P(x) \vee A(x)].$$

II) John's gold fish is not a plant.
 $\sim P(J)$

III) All animals have heart.
 $\forall (x) [A(x) \rightarrow H(x)]$

IV) John's gold fish has a heart.
 $H(J)$

The premises are

$$\forall x [P(x) \vee A(x)]$$

$$\sim P(J)$$

$$\forall x [A(x) \rightarrow H(x)]$$

$$\therefore H(J)$$

$$\{1\} (1) \forall x [P(x) \vee A(x)] \text{ Rule-P}$$

$$\{1\} (2) P(J) \vee A(J) \text{ Rule-VS}$$

$$\{3\} (3) \sim P(J) \text{ Rule-P}$$

$$\{1, 3\} (4) A(J) \text{ Rule-T (Implication E)}$$

$$\{5\} (5) \forall x [A(x) \rightarrow H(x)] \text{ Rule-P}$$

$$\{5\} (6) A(J) \rightarrow H(J) \text{ Rule-VS}$$

$$\{1, 3, 5\} (7) H(J) \text{ Rule-T (P, P} \rightarrow Q \equiv Q)$$

2) Verify the validity of following argument.

1) All men are mortal

2) Socrates is a man.

3) Therefore Socrates is a mortal

$H(x)$: x is a man

$M(x)$: x is a mortal

S : Socrates



Symbolic:

$$\forall (x) [H(x) \rightarrow M(x)].$$

$H(s)$

$M(s)$

$\therefore$ the premises are.

$$\forall (x) [H(x) \rightarrow M(x)].$$

$H(s)$

$M(s).$

$$\{1\} (1) \forall (x) [H(x) \rightarrow M(x)] \quad \text{Rule-P}$$

$$\{1\} (2) H(s) \rightarrow M(s) \quad \text{Rule-Us}$$

$$\{3\} (3) H(s) \quad \text{Rule-P}$$

$$\{1, 3\} (4) M(s) \quad \text{Rule-T}$$

* Predicate Calculus:

The logic based on the analysis of predicate is called predicate logic or predicate calculus.

Note: 1) Connectives can be used to form the compound statement from the given statement.

2) If S is an n place predicate & $a_1, a_2, a_3, a_4, \dots, a_n$ are the names of the objects then what is the predicate?

$S(a_1, a_2, a_3, \dots, a_n)$ is predicate statement

$\Rightarrow$ Compound Statement in predicate Calculus or logic:

1) Rama is a teacher & her teaching is good.

$$T(x) \wedge G(t)$$

2) Rama is a teacher or her teaching is good.

$$T(x) \vee G(t)$$

3) Rama is a teacher then her teaching is good.

$$T(x) \rightarrow G(t).$$

$\Rightarrow$ Free variables:

An occurrence of a variable that is not bounded by quantifier (either a universal quantifier, existential

quantifier) is said to be a free variable.

⇒ Bound variables:-

or occurrence of a variable that is bounded by a quantifier (either universal or existential quantifier) is said to be a bound variable.

⇒ Scope of the Quantifiers:-

→ the part of the logical expression or predicate formula to which a quantifier is applied is called the Scope of the quantifier.

→ Scope of the quantifier is the formula immediately followed by quantifiers.

Predicate formula	Bound variable(s)	Free Variable (F)	Scope of Quantifier (S)
1) $\forall x P(x, y)$	x	y	$P(x, y)$
2) $\forall x [P(x) \rightarrow Q(x)]$	x	-	$P(x) \rightarrow Q(x)$
3) $\forall x [(P(x) \rightarrow \exists y) R(x, y)]$	x, y	-	$\forall x: [P(x) \rightarrow \exists y R(x, y)]$
4) $\forall x [P(x) \wedge Q(x)]$	x	-	$\exists y: R(x, y)$ $P(x) \wedge Q(x)$
5) $\exists x [P(x) \wedge Q(x)]$	x	-	$P(x) \wedge Q(x)$
6) $\exists x P(x) \wedge Q(x)$	first - x	Second - x	$P(x) \wedge Q(x)$

✶ Predicates:-

Predicate is an expression of one or more variables defined on some specified domain.

(or)
It is a sentence that contains a finite no. of variables. It becomes a proposition when specific values are substituted for the variables.

→ Predicates has 2 parts in that one is variable / object / subject & second is

predicate.

- Predicate is denoted with capital letter
& variable is small letter.
Ex: Rama is a bachelor.
 $B(s)$.

1) 1-place predicate:-

If there is only 1 object/variable is associated with 1 predicate is called 1-place predicate.

Ex: x is an even number.
 $E(x)$.

2) 2-place predicate

If there are 2 objects/variables associated with 1 predicate is called 2-place predicate.

Ex: x is greater than y .
 $G(x, y)$.

3) m-place predicate:-

If there are m objects/variables is associated with one predicate is called m -place predicate.

Ex: Rama sits between Ram & Renu
 $S(x, y, z)$.

1) Ex: Let $P(x)$ denotes the statements $x > 3$, what are the truth values of $P(4)$ & $P(2)$

$$P(x): x > 3$$

$$P(4): 4 > 3 \rightarrow T$$

$$P(2): 2 > 3 \rightarrow F$$

2) Ex: Let $Q(x, y)$ denotes the statement $x = y + 3$, what are the truth values of proposition $Q(1, 2)$ & $Q(3, 0)$

$$x = y + 3$$

$$Q(1, 2): 2 = 1 + 3$$

$$2 = 5 \rightarrow F$$

$$Q(3, 0): 3 = 0 + 3$$

$$3 = 3 \rightarrow T$$

3) Let $P(x)$, x^2 is a +ve number for every real number x , then what

are truth values of x .

$$p(x) = x^2$$

$$p(-1) : (-1)^2 = 1 \rightarrow T$$

$$p(-2) : (-2)^2 = 4 \rightarrow T$$

* Quantifiers

Quantifiers are the word that refers to quantifier such as all. It tells for how many elements in a given predicate is true.

→ Quantifier are used to express quantifier without giving exact number.

Ex:- all, some, many, few, none etc.

Ex:- Can I have some water.

It consists of 2 types.

1) Universal Quantifier.

2) Existential Quantifier.

1) Universal Quantifier

The quantifier 'all' is said to be universal quantifier & it is denoted by $\forall$.

Ex:- Let $p(x)$ be a statement then $x+1 > x$
what are the truth values for $p(1)$, $p(2)$

$$p(x) : x+1 > x$$

$$p(1) : 1+1 > 1$$

$$2 > 1$$

$$p(2) : 2+1 > 2$$

$$3 > 2$$

the truth value is true.

$\therefore p(x)$ is true for all the integers $\forall x p(x)$.

Ex:- Every living thing is plant or animal

$$p(x) : x \text{ is a plant}$$

$$A(x) : x \text{ is a animal}$$

$$\forall x [p(x) \vee A(x)]$$

2) Existential Quantifier:

the quantifier 'Some' is called existential quantifier & it is denoted by $\exists$.

Ex: let $Q(x)$ be a statement then $x \in$ what are the truth values for

$Q(1), Q(2), Q(3)$

$Q(x): x < 2$

$Q(1): 1 < 2$

$Q(2): 2 < 2$

$Q(3): 3 < 2$

$\therefore Q(x)$ is true for only $Q(1) \Rightarrow x=1$
 $\exists x Q(x)$.

Ex: Some monkeys have tail.

$M(x): x$ is monkey

$T(x): x$ is tail

$\exists x [M(x) \rightarrow T(x)]$

i) For the universe of all integers
 let $P(x): x > 0$ - true integer.

$Q(x): x$ is an even or odd.

$R(x): x$ is perfect square

$S(x): x$ is divisible by 3

$T(x): x$ is divisible by 7.

Write down the following quantifier statements in symbolic form.

i) Atleast one integer is even.
 $\exists x Q(x)$

ii) there exists a true integer & that is even
 $\exists x [P(x) \wedge Q(x)]$

iii) Some even integers are divisible by 3.
 $\exists x (Q(x) \rightarrow S(x))$

iv) Every integer is either even or odd
 $\forall x Q(x)$

v) If x is an even & perfect square then x is not divisible by 3
 $\forall x [(Q(x) \wedge R(x)) \rightarrow \sim S(x)]$

* Logical Equivalence for Quantifiers

Two quantified statements are said to be logically equivalent whenever they have the same truth value in all possible situations.

Ex: $\forall x [p(x) \wedge q(x)]$

$[\forall x p(x)] \wedge [\forall x q(x)]$

Ex: $\forall x [p(x) \wedge q(x)] \Leftrightarrow [\forall x p(x)] \wedge [\forall x q(x)]$

1) $\exists x [p(x) \vee q(x)] \Leftrightarrow [\exists x p(x)] \vee [\exists x q(x)]$

2) $\exists x [p(x) \rightarrow q(x)] \Leftrightarrow [\exists x \sim p(x)] \vee [\exists x q(x)]$

→ Negation of the quantifier.

Rule for negation of the quantifier is change the universal to existential and vice versa.

Ex: $\sim [\forall x, p(x)] \equiv \exists x [\sim p(x)]$

Ex: $\sim [\exists x, p(x)] \equiv \forall x [\sim p(x)]$

$[\forall x (p(x) \vee q(x))] \wedge [\forall x (p(x) \wedge q(x))]$

i) Negation of each given statement
Simplify each of the following
given propositions

i) $\exists x [(p(x) \vee q(x))] \equiv \forall x [\sim p(x) \wedge \sim q(x)]$

ii) $\forall x [(p(x) \wedge \sim q(x))] \equiv \exists x [\sim p(x) \vee q(x)]$

iii) $\forall x [(p(x) \rightarrow q(x))] \equiv \exists x [p(x) \wedge \sim q(x)]$

iv) $\exists x [(p(x) \vee q(x)) \rightarrow r(x)] \equiv \forall x [(p(x) \vee q(x)) \wedge \sim r(x)]$