11/7/22

UNIT- III

NORMAL DISTRIBUTION

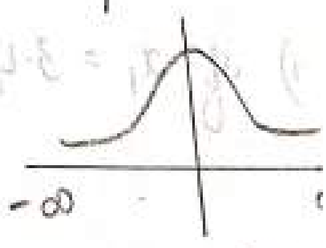
→ Normal distribution is a continuous probability distribution in which random variable takes infinite no. of values.

Def: A random variable X is said to have a normal distribution if its probability density function is defined as $f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where $-\infty < x, \mu < \infty$ and $\sigma > 0$

Here μ is mean and σ is standard deviation of normal distribution

chief characteristics (or) Properties of a Normal distribution

1. Normal probability curve (normal distribution) is obtained by $Z = \frac{x-\mu}{\sigma}$ and it is in bell shape



$Z = 0$
$x = \mu$

2. The total area under the normal probability curve is 1

3. Normal probability curve is symmetric with respect to $Z=0$ (or) $x=\mu$

4. The mean, median and mode of a normal distribution are equal

5. The area from $Z=0$ to $-\infty$ and 0 to ∞ are equal

6. If $x \in (a,b) \Rightarrow P(a<x<b) = \int_a^b f(x)\,dx$

1. For a normally distributed variate with mean 1, SD '3'. find the probability that

i) $3.43 \le x \le 6.19$

ii) $-1.43 \le x \le 6.19$

Given that $\mu = 1$, $\sigma = 3$

we know that $Z = \frac{x-\mu}{\sigma}$

i) if $x_1 = 3.43 \Rightarrow Z_1 = \frac{3.43-1}{3} = 0.81$

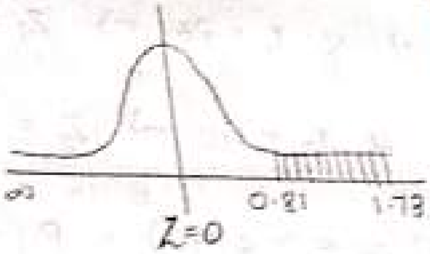
||ly if $x_2 = 6.19 \Rightarrow Z_2 = \frac{6.19-1}{3} = 1.73$

$P(3.43 \le x \le 6.19) = P(0.81 \le z \le 1.73)$

$= \phi(1.73) - \phi(0.81)$

$= 0.4582 - 0.2910$

$= 0.1672$



ii) if $x_1 = -1.43 \Rightarrow Z_1 = \frac{-1.43-1}{3} = -0.81$

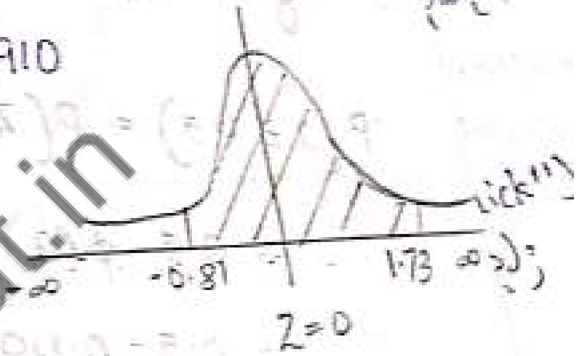
if $x_2 = 6.19 \Rightarrow Z_2 = \frac{6.19-1}{3} = 1.73$

$P(1.43 \le x \le 6.19) = P(-0.81 \le x \le 1.73)$

$= \phi(1.73) + \phi(0.81)$

$= 0.4582 + 0.2910$

$= 0.7492$



2. If x is a normal variant with mean 30 & SD 5, find the probabilities that

i) $26 \le x \le 40$

ii) $x \ge 45$

Given that $\mu = 30$, $\sigma = 5$

wkt $Z = \frac{x-\mu}{\sigma}$

i) if $x_1 = 26 \Rightarrow Z_1 = \frac{26-30}{5} = -0.8$

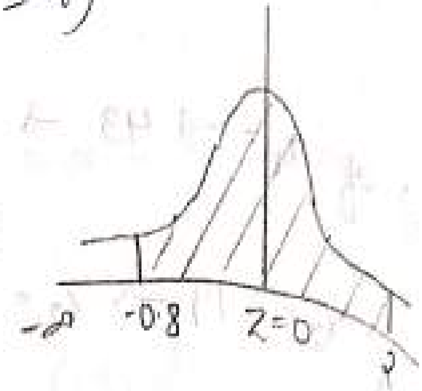
if $x_2 = 40 \Rightarrow Z_2 = \frac{40-30}{5} = 2$

$P(26 \le x \le 40) = P(-0.8 \le Z \le 2)$

$= \phi(2) + \phi(0.8)$

$= 0.4772 + 0.2881$

$= 0.7653$



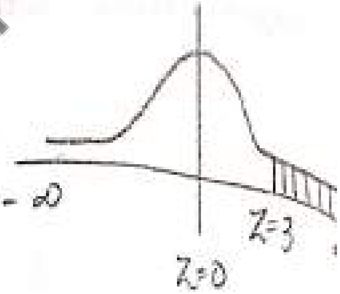
ii) if $x = 45 \Rightarrow Z = \frac{45-30}{5} = 3$

$P(x \ge 45) = P(Z \ge 3)$

$= 0.5 - \phi(3)$

$= 0.5 - 0.4987$

$= 0.0013$



3. If x is a normal variate find the area

i) to the left of $Z = -1.78$

ii) to the right of $Z = -1.45$

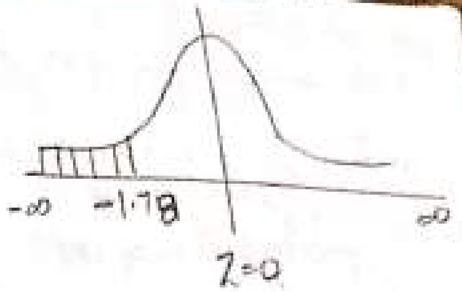
iii) corresponding to $-0.8 \le Z \le 1.58$

iv) to the left of $Z = -2.52$ & to the right of $Z = 1.83$

i) $0.5 = -\phi(1.78)$

$= 0.5 - 0.4625$

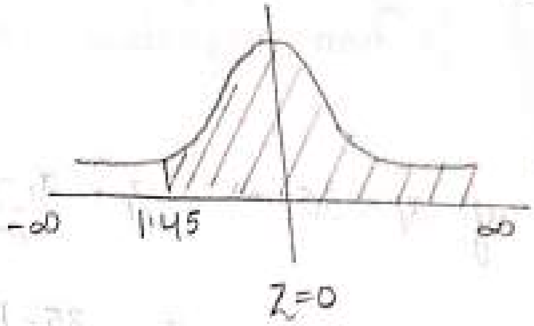
$= 0.0375$



ii) $= 0.5 + \phi(1.45)$

$= 0.5 + 0.4265$

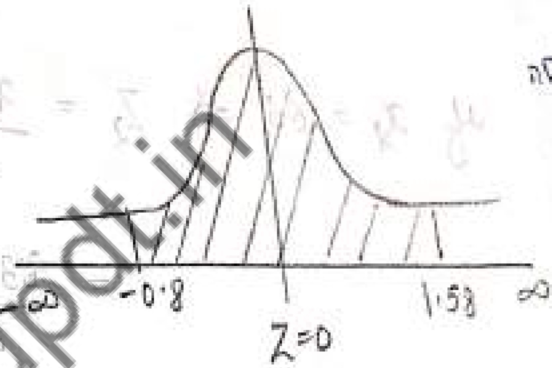
$= 0.9265$



iii) $= \phi(0.8) + \phi(1.58)$

$= 0.2881 + 0.4430$

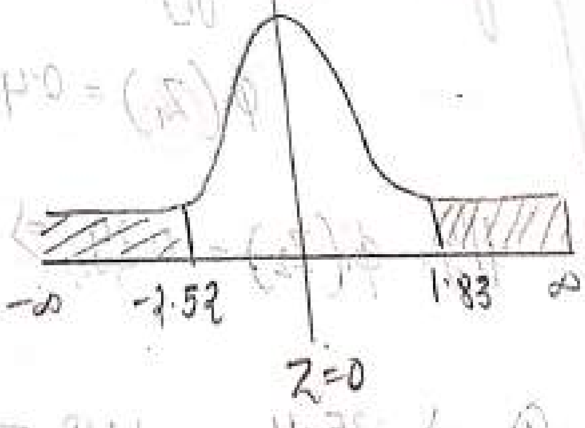
$= 0.7311$



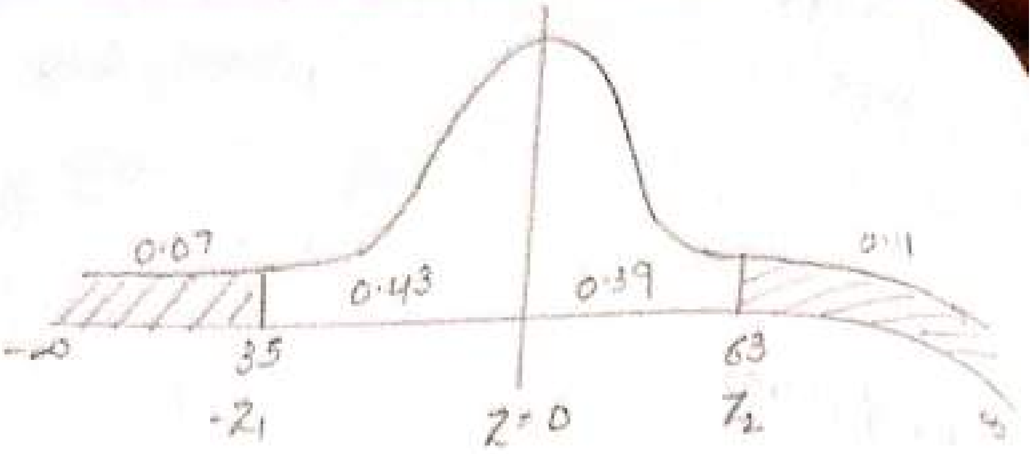
iv) $= 1 - \phi(2.52) - \phi(1.83)$

$= 1 - 0.4941 - 0.4664$

$= 0.0395$



4. In a normal distribution, 7% of items are under 35 & 89% of items are under 63 then find mean and variance of the distribution



if $x_1 = 35 \Rightarrow -Z_1 = \frac{x_1 - \mu}{\sigma}$

$$-Z_1 = \frac{35-\mu}{\sigma} \quad — (1)$$

if $x_2 = 63 \Rightarrow Z_2 = \frac{x_2 - \mu}{\sigma}$

$$= \frac{63-\mu}{\sigma} \quad — (2)$$

from the figure, we have

$$\phi(Z_1) = 0.43 \Rightarrow Z_1 = 1.48$$

$|||^{ly}$ $\phi(Z_2) = 0.39 \Rightarrow Z_2 = 1.23$

$(1) \Rightarrow \frac{35-\mu}{\sigma} = -1.48 \quad — (3)$

$(2) \Rightarrow \frac{63-\mu}{\sigma} = 1.23 \quad — (4)$

$\frac{(3)}{(4)} \Rightarrow \frac{35-\mu}{\sigma} \times \frac{\sigma}{63-\mu} = \frac{-1.48}{1.23}$

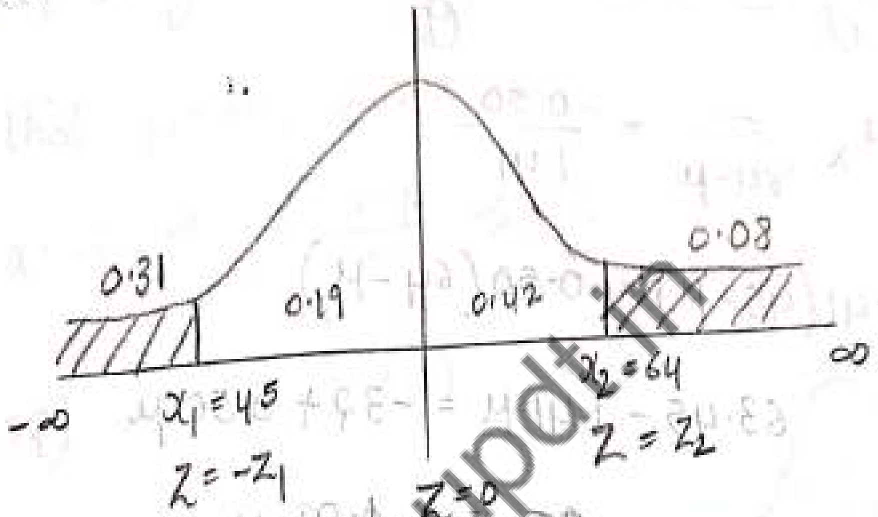
$\Rightarrow \mu = 50.29$ sub μ in eq (3) $\sigma = 10.3$

$\therefore \sigma^2 = 106.09$

18) 6/33

* In a Normal distribution 31% of items are under 45 and 8% of items are over 64. Then Find mean and variance of the distribution

Given that 31% of items are under 45 and 8% of items are over 64.



if $x_1 = 45 \Rightarrow -Z_1 = \frac{x_1 - \mu}{\sigma}$

$\Rightarrow -Z_1 = \frac{45 - \mu}{\sigma}$ — ①

if $x_2 = 64 \Rightarrow Z_2 = \frac{x_2 - \mu}{\sigma}$

$\Rightarrow Z_2 = \frac{64 - \mu}{\sigma}$ — ②

from the table $\phi(Z_1) = 0.19$

$\Rightarrow Z_1 = 0.50$

IIIly $\phi(z_2) = 0.42$

$\Rightarrow z_2 = 1.41$

① $\Rightarrow \dfrac{45-\mu}{\sigma} = -0.50$ — ③

② $\Rightarrow \dfrac{64-\mu}{\sigma} = 1.41$ — ④

by solving ③ & ④ $\Rightarrow \dfrac{③}{④}$

$$\frac{45-\mu}{\sigma} \times \frac{\sigma}{64-\mu} = \frac{-0.50}{1.41}$$

$$1.41(45-\mu) = -0.50(64-\mu)$$

$$63.45 - 1.41\mu = -32 + 0.50\mu$$

$$95.45 = 1.91\mu$$

$$\mu = 49.97 \simeq 50$$

sub μ in eqv ③

$\cancel{\sigma = \dfrac{45-49.97}{-0.50} = 9.94}$

$$\sigma = \frac{45-50}{0.50} = 10$$

mean of the distribution, $\mu = 50$ and

variance of the distribution, $\sigma^2 = 100$

[illegible] If the masses of 300 students are normally distributed with mean 68 kgs & standard deviation 3 kgs. How many students have masses

i) greater than 72 kg

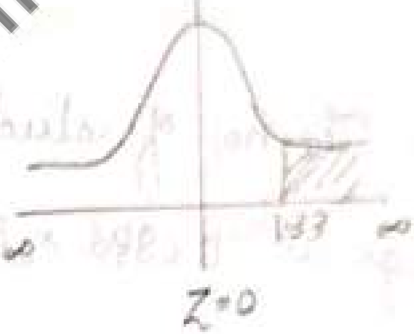
ii) less than or equal to 64 kg

iii) between 65 kg and 71 kg inclusive

Given that $\mu = 68$, $\sigma = 3$

i) if $x = 72 \Rightarrow Z = \frac{x-\mu}{\sigma} \Rightarrow Z = \frac{72-68}{3} = 1.33$

$P(x > 72) = P(Z > 1.33)$



$\Rightarrow 0.5 - \phi(1.33)$

$0.5 - 0.4082$

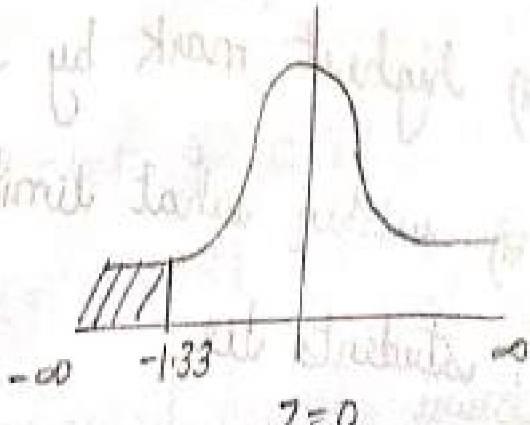
$= 0.0918$

$\therefore$ The no. of students whose mass is >72kgs is

$300 \times 0.0918 = 27.54 \simeq 28$

ii) if $x = 64 \Rightarrow Z = \frac{64-68}{3} = -1.33$

$\Rightarrow P(x \leq 64) = P(Z \leq -1.33)$



$\Rightarrow 0.5 - \phi(1.33)$

$0.5 - 0.4082$

$= 0.0918$

$\therefore$ The no. of students whose mass is ≤ 64 kg

$300 \times 0.0918 = 27.54 \cong 28$

iii) if $x_1 = 65 \Rightarrow z_1 = \frac{65-68}{3} = -1$

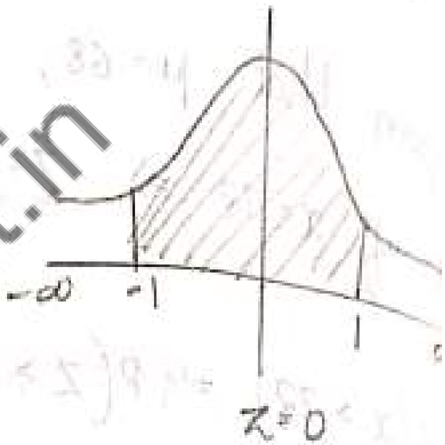
if $x_2 = 71 \Rightarrow z_2 = \frac{71-68}{3} = 1$

$\Rightarrow P(65 \leq x \leq 71) = P(-1 \leq z \leq 1)$

$\Rightarrow \phi(1) + \phi(1)$

$2\phi(1)$

$2(0.3413) = 0.6826$



$\therefore$ The no. of students whose masses b/w 65 & 71 kgs is $0.6826 \times 300 = 204.78 \cong 205$

* The marks obtained in mathematics by 1000 students is normally distributed with mean 78% & SD 11%. Then determine

i) how many students got marks above 90%

ii) highest mark by lowest 10% of students

iii) within what limits did the middle of 90% of the students lie

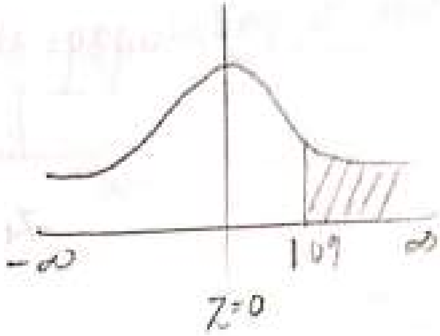
Given that $\mu = 78\% \Rightarrow \mu = 0.78$

$\sigma = 11\% \Rightarrow \sigma = 0.11$

i) if $x = 90\% \Rightarrow x = 0.9 \Rightarrow Z = \frac{x-\mu}{\sigma}$

$$\Rightarrow Z = \frac{0.9-0.78}{0.11} = 1.09$$

i.e. $P(x > 0.9) = P(Z > 1.09)$



$\Rightarrow 0.5 - \phi(1.09)$

$0.5 - 0.3621$

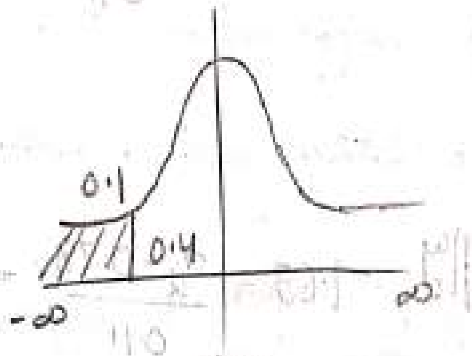
$= 0.1379$

$\therefore$ The no. of students whose marks is $> 90\%$ is

$0.1379 \times 1000 = 137.9 \cong 138$

ii) the lowest 10% constitutes the left side of

$Z = 0.1$



Here $\phi(Z_1) = 0.4$

$\Rightarrow Z_1 = -1.29$

$\therefore Z = \frac{x-\mu}{\sigma}$

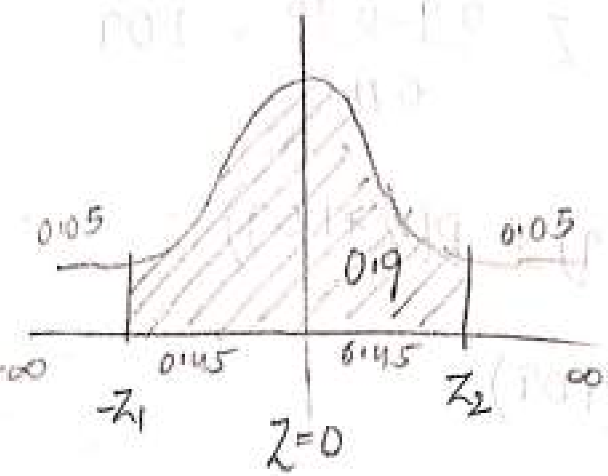
$\Rightarrow -1.29 = \frac{x - 0.78}{0.11} \Rightarrow x = -0.1419 + 0.78$

$x = 0.6381 = 63.81\%$

i.e. 64 is the highest mark obtained by the lowest

10% of students

iii) 90% area constitute 0.9 under the curve i.e. leaving 0.05 area ~~and~~ on both sides we have



Here $\phi(z_1) = 0.45 \Rightarrow z_1 = -1.65$

IIly $\phi(z_2) = 0.45 \Rightarrow z_2 = 1.65$

we have $z = \frac{x-\mu}{\sigma}$

$\Rightarrow -1.65 = \frac{x_1 - 0.78}{0.11} \Rightarrow x_1 = -0.1815 + 0.78$

$x_1 = 0.5985 = 59.85\%$

IIly $1.65 = \frac{x_2 - 0.78}{0.11} \rightarrow x_2 = 0.1815 + 0.78$

$= 0.9615 = 96.15\%$

$\therefore$ middle of the 90% students lie b/w 60 & 96 marks.

23/6/23 Sampling Distribution

Population

The total no. of objects in the system is known as a population. If the no. of objects in the population are finite, it is known as finite population or else it is infinite population.

Population Size

no. of objects in the population is known as population size. It is represented by 'N'

Sample

A small portion which is taken from population is known as sample.

Sample size

No. of observations in the sample is known as sample size. It is represented by 'n'.

If the sample size $n \le 30$, that is known as small sample.

If $n > 30$, that is known as a large sample

Parameter

Population measures such as mean, variance, standard deviation are known as parameters.

Statistic

Sample measures such as mean, variance, ... are known as Statistic.

Sampling

The method of choosing sample from the population is known as sampling. The probability Distribution of statistic is known as sampling distribution.

There are 2 types of sampling distribution

i) Sample with replacement (n^r)

ii) Sample without replacement $({}^nC_r)$ $(r \leq n)$

Formulas

1) If $x_1, x_2, x_3 \ldots x_n$ performs a population then

i) population mean 'μ' $= \sum_{i=1}^{n} \frac{x}{N}$

$\mu = \frac{x_1 + x_2 + \ldots x_n}{}$

ii) population variance

$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N} \Rightarrow \frac{(x_1-\mu)^2 + (x_2-\mu)^2 + \cdots (x_n-\mu)^2}{N}$$

iii) population S.D

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{N}} \Rightarrow \sqrt{\frac{(x_1-\mu)^2 + (x_2-\mu)^2 + \cdots (x_n-\mu)^2}{N}}$$

If $x_1, x_2, \ldots x_n$ performs a sample, then

i) Sample mean

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \Rightarrow \frac{x_1 + x_2 + \cdots + x_n}{n}$$

ii) Sample Variance

$$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1} \Rightarrow \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2}{n-1}$$

iii) Sample S.D.

$$S = \sqrt{\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{n-1}}$$

* The standard Error of sample mean is $\frac{\sigma}{\sqrt{n}}$

* Standard Error of sample S.D is $\frac{\sigma}{\sqrt{2n}}$

* Standard Error of sample proportion $\sqrt{\frac{PQ}{n}}$

* Finite Population correction factor is $\frac{N-n}{N-1}$

27/6/29 ** ★ A population consisting of 5 numbers 2, 3, 6, 8, 11. Consider all the possible samples of size 2 that can be taken from the given population with replacement. Hence, find

i) mean of the population

ii) S.D of the population

iii) mean of the sampling distribution of means

iv) S.D of the sampling distribution of means

Given that $N = 5$, $r = 2$

wkt no. of samples of size 2 taken from given population with replacement is $n^r = 5^2 = 25$

$\Rightarrow$ 25 samples are

(2,2), (2,3), (2,6), (2,8), (2,11)

(3,2), (3,3), (3,6), (3,8), (3,11)

(6,2), (6,3), (6,6), (6,8), (6,11)

(8,2), (8,3), (8,6), (8,8), (8,11)

(11,2), (11,3), (11,6), (11,8), (11,11)

i) wkt mean of the population

$\mu = \sum_{i=1}^{N} \frac{x_i}{N} \Rightarrow \frac{2+3+6+8+11}{5} = 6$

ii) S.D of the population

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \mu)^2}{N}} = \sqrt{\frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2}{5}}$$

$\sigma = 3.28$

iii) wkt mean of each sample is

2, 2.5, 4, 5, 6.5

2.5, 3, 4.5, 5.5, 7

4, 4.5, 6, 7, 8.5

5, 5.5, 7, 8, 9.5

6.5, 7, 8.5, 9.5, 11

The mean of the sampling distribution of means is

$$\frac{2 + 2.5 + 4 + \cdots + 9.5 + 11}{25} = \frac{150}{25} = 6$$

iv) S.D of the sampling distribution of means

$$\sqrt{\frac{(2-6)^2 + (2.5-6)^2 + \cdots + (11-6)^2}{25}} = 2.32$$

* List all possible samples of size 2 that can be taken from given population 2, 3, 6, 8, 11 without replacement. Hence, find

i) mean of the population

ii) S.D of the population

iii) mean of sampling distribution of means

iv) S.D of sampling distribution of means

Given that $N = 5$, $r = 2$

wkt no. of samples of size 2 taken from given population without replacement is ${}^{N}C_{r} = {}^{5}C_{2} = 10$

$\rightarrow$ 10 samples are

(2,3), (2,6), (2,8), (2,11)

(3,6), (3,8), (3,11)

(6,8), (6,11)

(8,11)

(1,5) (1,6) (1,8)

(5,6) (5,8)

(6,8)

i) wkt mean of the population

$$\mu = \sum_{i=1}^{N} \frac{x_i}{N} \Rightarrow \frac{2+3+6+8+11}{5} = 6$$

ii) S.D of the population

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(a_i-\mu)^2}{N}} \Rightarrow \sqrt{\frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5}}$$

$$\sigma = 3.28$$

iii) wkt mean of each sample is

2.5, 4, 5, 6.5

4.5, 5.5, 7

7, 8.5

9.5

The mean of sampling distribution of means is

$$\frac{2.5+4+\cdots+8.5+9.5}{10} = 6$$

iv) S.D of the sampling distribution of means is

$$\sqrt{\frac{(2.5-6)^2+(4-6)^2+\cdots(9.5-6)^2}{10}} = 2.01$$

* If the population is 3, 6, 9, 15, 27 then

i) list all possible samples of size 3 that can be taken from given population without replacement

ii) find the mean of the population

iii) S.D of the population

iv) mean of the sampling distribution of means

v) S.D of the sampling distribution of means

i) wkt the no. of samples of size 3 taken from given population is $^5C_3 = 10$

$\Rightarrow$ 10 samples are

(3,6,9), (3,6,15), (3,6,27), (3,15,27)

(3,9,15), (3,9,27)

(6,9,15), (6,9,27), (6,15,27), (9,15,27)

ii) mean of the population

$$\mu = \sum_{i=1}^{N} \frac{x_i}{N} = \frac{3+6+9+15+27}{5} = 12$$

iii) S.D of the population

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \mu)^2}{N}} = \sqrt{\frac{(3-12)^2+(6-12)^2+(9-12)^2+(15-12)^2+(27-12)^2}{5}}$$

$$\sigma = 8.48$$

iv) wkt mean of each sample is

6, 8, 12, 15, 9, 13, 10, 14, 16, 17

$$\frac{6+8+\cdots+16+17}{10} = 12$$

iv) S.D of the sampling distribution of means is

$$\sqrt{\frac{(6-12)^2+(8-12)^2+\cdots+(17-12)^2}{10}} = 3.46$$

28/6/22 <u>Central Limit Theorem</u>

If $\bar{x}$ is the random sample mean of sample size 'n' taken from a population having mean μ & S.D 'σ' then the central limit theorem states that $Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

* A random sample of size 100 is taken from an infinite population having the mean 76 and the variance 256, then what is the probability that sample mean $\bar{x}$ will be b/w 75 & 78.

Given that $n=100$, $\mu=76$, $\sigma^2=256 \Rightarrow \sigma=16$

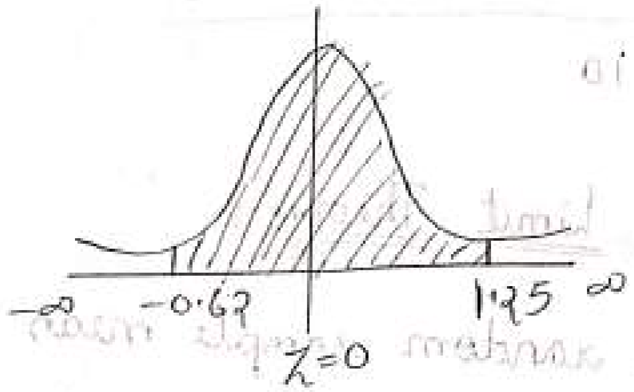
By Central limit theorem, we have

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

If $\bar{x}=75 \Rightarrow Z = \frac{75-76}{16} = -0.625$

|||ly if $\bar{x} = 78 \Rightarrow Z = \frac{78-76}{\frac{16}{10}} = \frac{20}{16} = 1.25$

$\Rightarrow P(75 \le \bar{x} \le 78) = P(-0.62 \le Z \le 1.25)$



$\Rightarrow \phi(0.62) + \phi(1.25)$

$= 0.2324 + 0.3944$

$= 0.6268$

* A Random sample of size 64 is taken from a normal population with mean 51.4 & S.D 6.8. what the probability that the mean of the sample will

i) exceed 52.9

ii) fall b/w 50.5 & 52.3,

iii) less than 50.6

Given that $n = 64$, $\mu = 51.4$, $\sigma = 68$

By Central limit theorem, we have $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

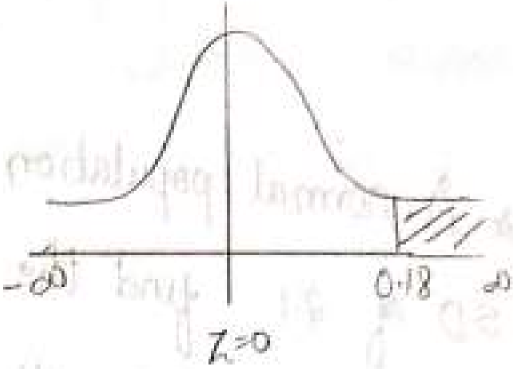
if $\bar{x} = 52.9 \Rightarrow Z = \dfrac{52.9 - 51.4}{\frac{68}{8}} = 0.18$

i) $\rightarrow P(\bar{x} > 52.9) = P(Z > 0.18)$

$\Rightarrow 0.5 - \phi(0.18)$

$= 0.5 - 0.0714$

$= 0.4286$



ii) if $\bar{x} = 50.5 \Rightarrow Z = \dfrac{50.5 - 51.4}{68/8} = -0.10$

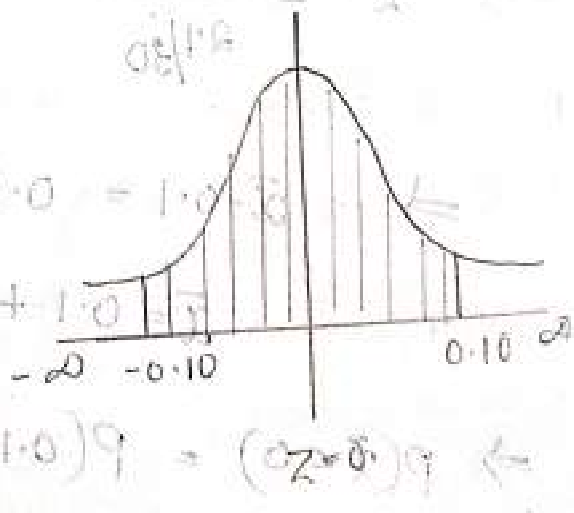
|||ly if $\bar{x} = 52.3 \Rightarrow Z = \dfrac{52.3 - 51.4}{68/8} = 0.10$

$\Rightarrow P(50.5 \leq \bar{x} \leq 52.3) = P(-0.10 \leq Z \leq 0.10)$

$\Rightarrow 2\phi(0.10)$

$= 2(0.0398)$

$= 0.0796$



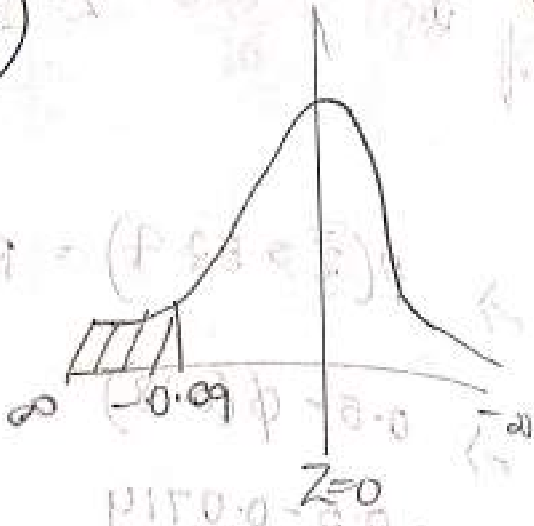
iii) if $\bar{x} = 50.6 \Rightarrow Z = \dfrac{50.6 - 51.4}{\frac{68}{8}} = -0.09$

$P(\bar{x} < 50.6) = P(Z < 0.09)$

$\Rightarrow 0.5 - \phi(0.09)$

$= 0.5 - 0.0359$

$= 0.4641$



★ A normal population has a mean of 0.1 & S.D of 2.1. find the probability that mean of the sample size 900 will be negative.

Given that $\mu = 0.1$, $\sigma = 2.1$, $n = 900$

By Central Limit Theorem we have,

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$\Rightarrow Z = \frac{\bar{x} - 0.1}{2.1/30} = \frac{\bar{x} - 0.1}{0.07}$$

$$\Rightarrow \bar{x} - 0.1 = 0.07(Z)$$

$$\bar{x} = 0.1 + 0.07(Z)$$

$$\Rightarrow P(\bar{x} < 0) = P(0.1 + 0.07(Z) < 0)$$

$$= P(0.07 Z < -0.1)$$

$$= P\left(Z < \frac{-0.1}{0.07}\right)$$

$= P(z < -1.43)$

$\Rightarrow 0.5 - \phi(1.43)$

$= 0.5 - 0.4236$

$= 0.0764$